Tensorial logic with algebraic effects

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Logic and interactions
Week 3 – Proofs and programs

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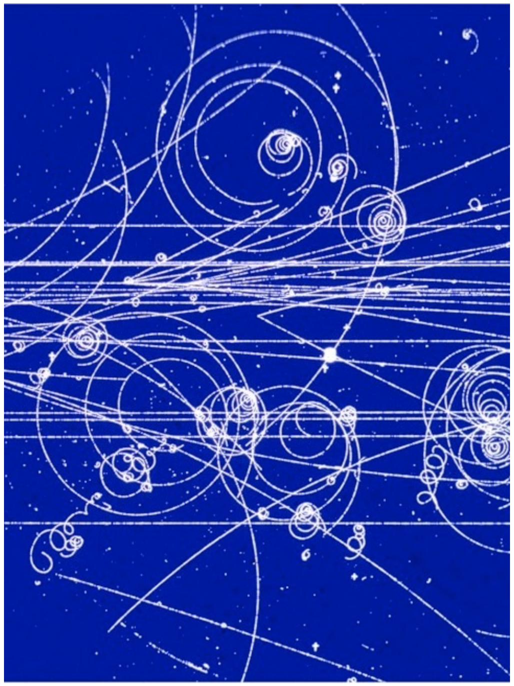
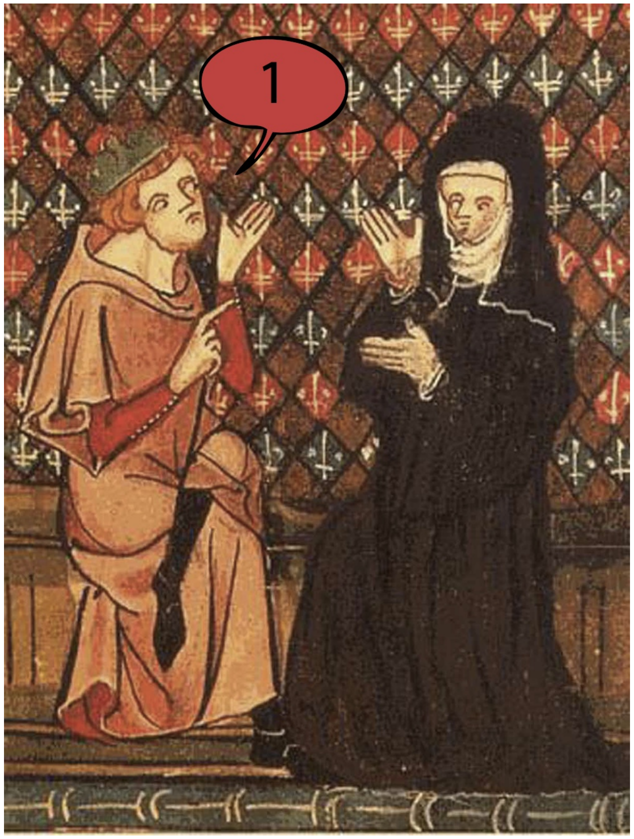
Logic

Physics



Like physics, logic should be the description of **a material event**...

The logical phenomenon

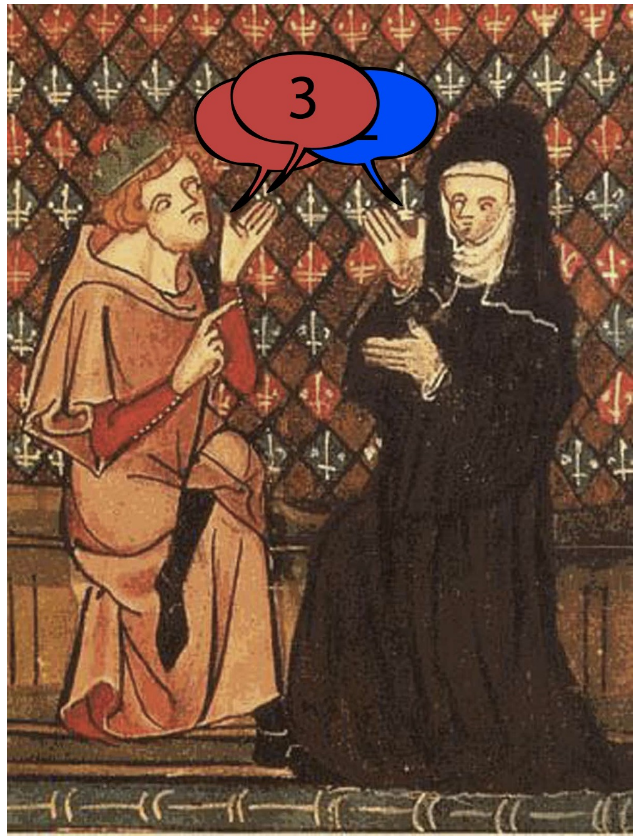


What is the topological structure of a dialogue?

The logical phenomenon

What is the topological structure of a dialogue?

The logical phenomenon



What is the topological structure of a dialogue?

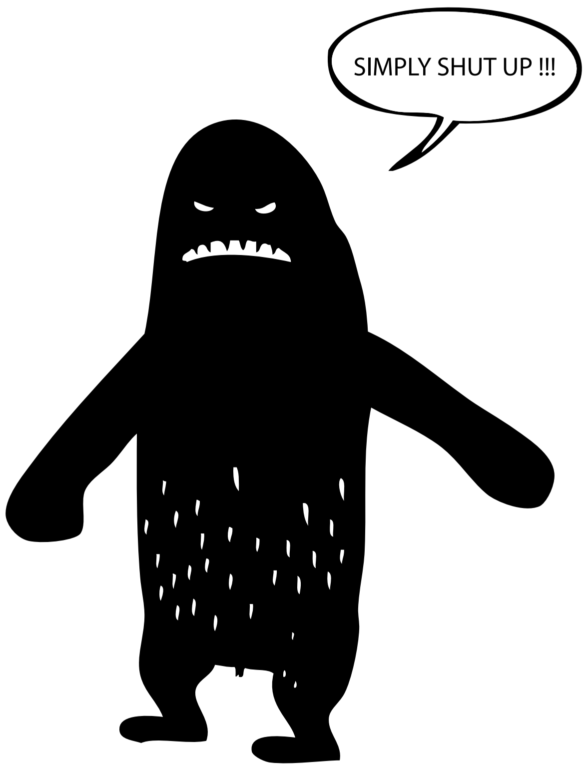
The basic symmetry of logic

The discourse of reason is **symmetric** between Player and Opponent

Claim: this symmetry is the foundation of logic

Next question: can we reconstruct logic from this basic symmetry?

The microcosm principle



No contradiction (thus no formal logic) can emerge in a tyranny...

A microcosm principle in algebra
[Baez & Dolan 1997]

The definition of a **monoid**

$$M \quad \times \quad M \qquad \longrightarrow \qquad M$$

requires the ability to define a **cartesian product** of sets

$$A \quad , \quad B \qquad \mapsto \qquad A \times B$$

Structure at dimension 0 requires structure at dimension 1

A microcosm principle in algebra
[Baez & Dolan 1997]

The definition of a **cartesian** category

$$\mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

requires the ability to define a **cartesian product** of categories

$$\mathscr{A}\ ,\ \mathscr{B} \mapsto \mathscr{A} \times \mathscr{B}$$

Structure at dimension 1 requires structure at dimension 2

A similar microcosm principle in logic

The definition of a cartesian **closed** category

$$\mathscr{C}^{op} \quad \times \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

requires the ability to define the **opposite** of a category

$$\mathscr{A} \quad \mapsto \quad \mathscr{A}^{op}$$

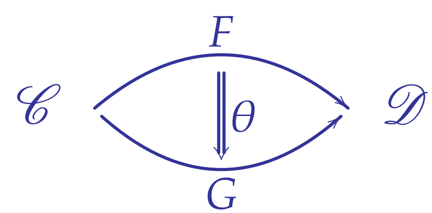
Hence, the “implication” at level 1 requires a “negation” at level 2

An automorphism in Cat

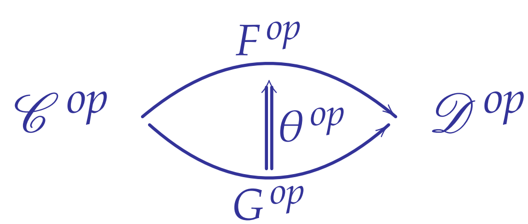
The 2-functor

$$op \quad : \quad \underline{Cat} \quad \longrightarrow \quad \underline{Cat}^{\,op(2)}$$

transports every natural transformation



to a natural transformation in the opposite direction:



$\longrightarrow$ requires a braiding on $\mathscr{V}$ in the case of $\mathscr{V}$-enriched categories

Chiralities

A symmetrized account of categories

From categories to chiralities

A slightly bizarre idea emerges in order to reflect the symmetry of logic:

decorrelate the category $\mathscr{C}$ from its opposite category $\mathscr{C}^{op}$

So, let us define a **chirality** as a pair of categories $(\mathscr{A}, \mathscr{B})$ such that

$$\mathscr{A} \quad \cong \quad \mathscr{C} \qquad\qquad \mathscr{B} \quad \cong \quad \mathscr{C}^{op}$$

for some category $\mathscr{C}$.

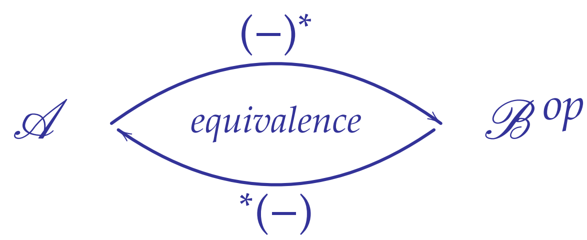
Here $\quad\cong\quad$ means **equivalence** of category

Chirality

More formally:

Definition:

A chirality is a pair of categories $(\mathscr{A}, \mathscr{B})$ equipped with an equivalence:



Chirality homomorphisms

Definition. A chirality homomorphism

$$(\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is a pair of functors

$$F_\bullet \quad : \quad \mathscr{A}_1 \quad \longrightarrow \quad \mathscr{A}_2 \qquad\qquad F_\circ \quad : \quad \mathscr{B}_1 \quad \longrightarrow \quad \mathscr{B}_2$$

equipped with a natural isomorphism

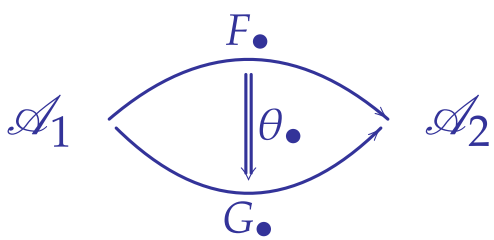
$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow{\quad F_\bullet \quad} & \mathscr{A}_2 \\ {\scriptstyle (-)^*}\downarrow & \widetilde{F} \;\Swarrow & \downarrow{\scriptstyle (-)^*} \\ \mathscr{B}_1^{\,op} & \xrightarrow[\quad F_\circ^{op} \quad]{} & \mathscr{B}_2^{\,op} \end{array}$$

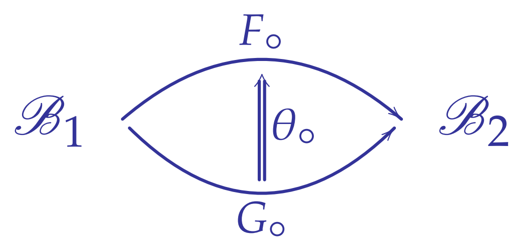
Chirality transformations

Definition. A chirality transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

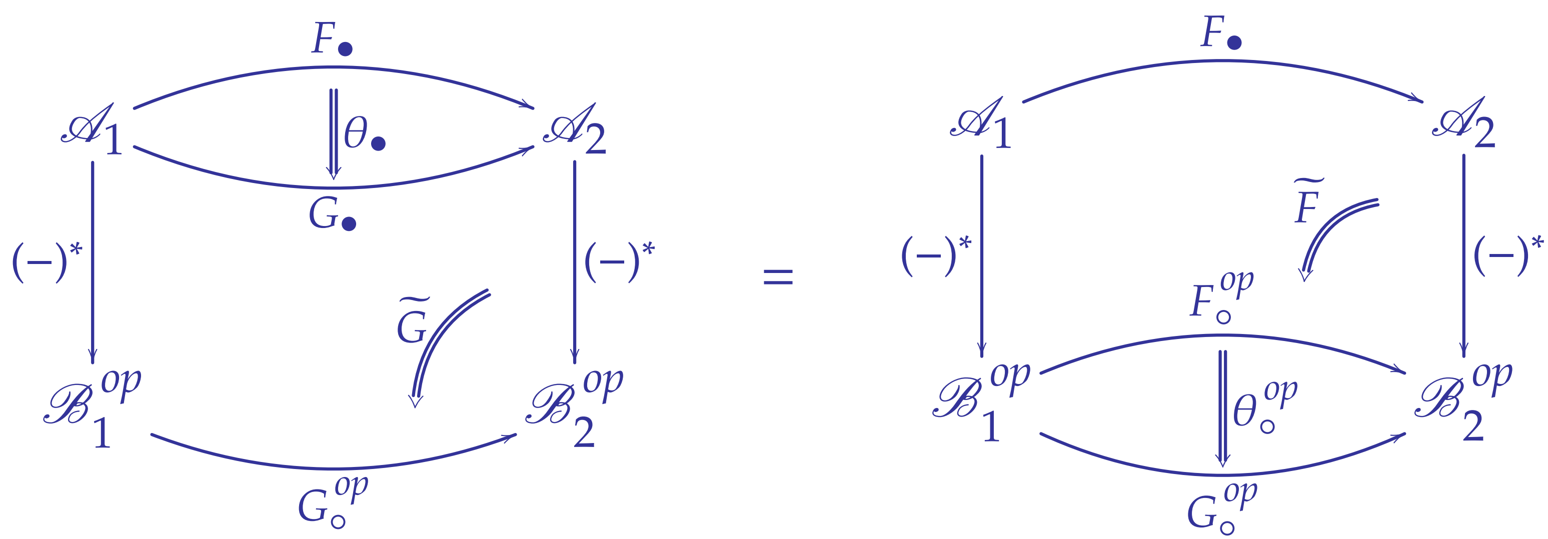
is a pair of natural transformations





Chirality transformations

satisfying the equality



A technical justification of symmetrization

Let *Chir* denote the 2-category with

- ▷ chiralities as objects
- ▷ chirality homomorphism as 1-dimensional cells
- ▷ chirality transformations as 2-dimensional cells

Proposition. The 2-category *Chir* is biequivalent to the 2-category *Cat*.

Cartesian closed chiralities

A symmetrized account of cartesian closed categories

Cartesian chiralities

Definition. A cartesian chirality is a chirality

- ▷ whose category $\mathscr{A}$ has finite products noted

$$a_1 \wedge a_2 \qquad\qquad \mathbf{true}$$

- ▷ whose category $\mathscr{B}$ has finite sums noted

$$b_1 \vee b_2 \qquad\qquad \mathbf{false}$$

Cartesian closed chiralities

Definition. A cartesian closed chirality is a cartesian chirality

$$(\mathscr{A}, \wedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \vee, \mathbf{false})$$

equipped with a pseudo-action

$$\vee \quad : \quad \mathscr{B} \quad \times \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

and a bijection

$$\mathscr{A}(a_1 \wedge a_2, a_3) \quad \cong \quad \mathscr{A}(a_1, a_2^* \vee a_3)$$

natural in a_1, a_2 and a_3.

Once symmetrized, the definition of a ccc becomes purely algebraic

Dictionary

The pseudo-action

$$\vee \quad : \quad \mathscr{B} \quad \times \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

reflects the functor

$$\Rightarrow \quad : \quad \mathscr{C}^{\,op} \quad \times \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

The isomorphisms defining the pseudo-action

$$(b_1 \vee b_2) \vee a \quad \cong \quad b_1 \vee (b_2 \vee a) \qquad\qquad \mathbf{false} \vee a \quad \cong \quad a$$

reflect the familiar isomorphisms

$$(x_1 \times x_2) \Rightarrow y \quad \cong \quad x_1 \Rightarrow (x_2 \Rightarrow y) \qquad\qquad 1 \Rightarrow x \quad \cong \quad x$$

Dictionary continued

The isomorphism

$$\mathscr{A}(a_1 \wedge a_2, a_3) \quad \cong \quad \mathscr{A}(a_2, a_1^* \vee a_3)$$

reflects the familiar isomorphism

$$\mathscr{A}(x \times y, z) \quad \cong \quad \mathscr{A}(y, x \Rightarrow z)$$

Note that the isomorphism

$$(a_1)^* \vee a_2 \qquad \cong \qquad a_1 \Rightarrow a_2$$

deserves the name of **classical decomposition** of the implication...
although we are in a cartesian closed category!

Dictionary continued

So, what distinguishes classical logic from intuitionistic logic...
are not the connectives themselves, but their algebraic structure.

Typically, the disjunction $\vee$ is:

- ▷ a pseudo-action in the case of cartesian closed chiralities,
- ▷ the cotensor product $\parr$ in the case of $*$-autonomous categories.

Cartesian closed chirality [in Krivine style]

Definition. A cartesian closed chirality is a chirality

$$(\mathscr{A}, \wedge, \mathbf{true}) \qquad \mathscr{B}$$

where $\mathscr{A}$ has finite products, equipped with a pseudo-action

$$\wedge \quad : \quad \mathscr{A} \quad \times \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{B}$$

and a bijection

$$\langle a_1 \wedge a_2, b \rangle \quad \cong \quad \langle a_1, a_2 \wedge b \rangle$$

natural in a_1, a_2 and b where

$$\langle a, b \rangle \quad = \quad \mathscr{A}(a, {}^*b)$$

Cartesian closed chirality [in Krivine style]

Definition. A cartesian closed chirality is a chirality

$$(\Lambda, \times, \mathbf{true}) \qquad \Pi$$

where Λ has finite products, equipped with a pseudo-action

$$\bullet \quad : \quad \Lambda \quad \times \quad \Pi \quad \longrightarrow \quad \Pi$$

and a bijection

$$\langle a_1 \times a_2, b \rangle \quad \cong \quad \langle a_1, a_2 \bullet b \rangle$$

natural in a_1, a_2 and b, where

$$\langle a, b \rangle \quad = \quad \Lambda(a, {}^*b).$$

Stack is the hacker's name for action

Given a pseudo-action

$$\bullet \quad : \quad \Lambda \quad \times \quad \Pi \quad \longrightarrow \quad \Pi$$

every object of the category Π

$$b$$

every sequence of objects of the category Λ

$$a_1 \quad , \quad \ldots \quad , \quad a_n$$

define an object of the category Π

$$a_1 \bullet \ldots \bullet a_n \bullet b$$

Here, the object b should be seen as the bottom of the stack

A few observations

- ▷ Symmetrization does not need classical logic
- ▷ After symmetrization, the definition of a ccc becomes “algebraic”
- ▷ The equivalence $\mathscr{A} \cong \mathscr{B}^{op}$ may be easily relaxed.

Dialogue categories

A type-theoretic approach to game semantics

Tensorial logic

tensorial logic	=	a logic of tensor and negation
	=	linear logic without $A \cong \neg\neg A$
	=	the syntax of linear continuations
	=	the syntax of dialogue games

A tentative synthesis of linear logic and game semantics

Motivation: think of Guy's dialogue strategies as tensorial proof-nets

Six primitive components of logic

[1]	the negation	$\neg$
[2]	the linear conjunction	$\otimes$
[3]	the sum	$\oplus$
[4]	the repetition modality	$!$
[5]	the existential quantification	$\exists$
[6]	the least fixpoint	μ

Logic = Data Structure + Duality

Tensorial logic

Axiom	$\dfrac{}{A \vdash A}$	$\dfrac{\Gamma \vdash A \qquad \Upsilon_1, A, \Upsilon_2 \vdash B}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B}$	Cut
Right $\otimes$	$\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$	$\dfrac{\Upsilon_1, A, B, \Upsilon_2 \vdash C}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash C}$	Left $\otimes$
Right I	$\dfrac{}{\vdash I}$	$\dfrac{\Upsilon_1, \Upsilon_2 \vdash A}{\Upsilon_1, I, \Upsilon_2 \vdash A}$	Left I

Tensorial logic

Right $\multimapinv$ $$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \bot \multimapinv A}$$

$$\frac{\Gamma \vdash A}{\bot \multimapinv A, \Gamma \vdash \bot}$$ Left $\multimapinv$

Right $\multimap$ $$\frac{A, \Gamma \vdash \bot}{\Gamma \vdash A \multimap \bot}$$

$$\frac{\Gamma \vdash A}{\Gamma, A \multimap \bot \vdash \bot}$$ Left $\multimap$

A primitive kernel of logic

Resource modalities and quantification

Weakening
$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$
Contraction

Promotion
$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$
Dereliction

Left ∃
$$\frac{\Gamma, A(x) \vdash B}{\Gamma, \exists x.A \vdash B}$$

$$\frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x.A}$$
Right ∃

Dialogue categories

A **monoidal category** with a **left duality**

A natural bijection between the set of maps

$$A \otimes B \quad \longrightarrow \quad \bot$$

and the set of maps

$$B \quad \longrightarrow \quad A \multimap \bot$$

A familiar situation in tensorial algebra

Dialogue categories

A **monoidal category** with a **right duality**

A natural bijection between the set of maps

$$A \otimes B \quad \longrightarrow \quad \bot$$

and the set of maps

$$A \quad \longrightarrow \quad \bot \multimapinv B$$

A familiar situation in tensorial algebra

Dialogue categories

Definition. A dialogue category is a monoidal category $\mathscr{C}$ equipped with

▷ an object $\bot$

▷ two natural bijections

$$\varphi_{A,B} \quad : \quad \mathscr{C}(A \otimes B, \bot) \quad \longrightarrow \quad \mathscr{C}(B, A \multimap \bot)$$

$$\psi_{A,B} \quad : \quad \mathscr{C}(A \otimes B, \bot) \quad \longrightarrow \quad \mathscr{C}(A, \bot \mathrel{\circ\!\!-} B)$$

Helical dialogue categories

A dialogue category equipped with a family of bijections

$$\mathit{wheel}_{A,B} \quad : \quad \mathscr{C}(A \otimes B, \bot) \quad \longrightarrow \quad \mathscr{C}(B \otimes A, \bot)$$

natural in A and B making the diagram

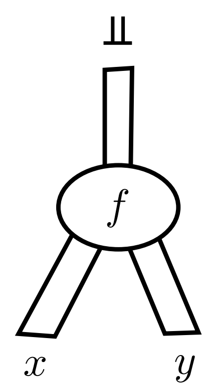
$$\begin{array}{ccc}
\mathscr{C}((B \otimes C) \otimes A, \bot) & \xrightarrow{\mathit{associativity}} & \mathscr{C}(A \otimes (C \otimes B), \bot) \\
\uparrow \mathit{wheel}_{A,B\otimes C} & & \downarrow \mathit{wheel}_{B,C\otimes A} \\
\mathscr{C}(A \otimes (B \otimes C)) & & \mathscr{C}((C \otimes A) \otimes B, \bot) \\
\downarrow \mathit{associativity} & & \uparrow \mathit{associativity} \\
\mathscr{C}((A \otimes B) \otimes C, \bot) & \xrightarrow{\mathit{wheel}_{A\otimes B,C}} & \mathscr{C}(C \otimes (A \otimes B), \bot)
\end{array}$$

commutes.

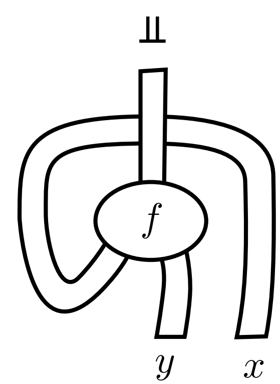
Helical dialogue categories

The wheel should be understood diagrammatically as:

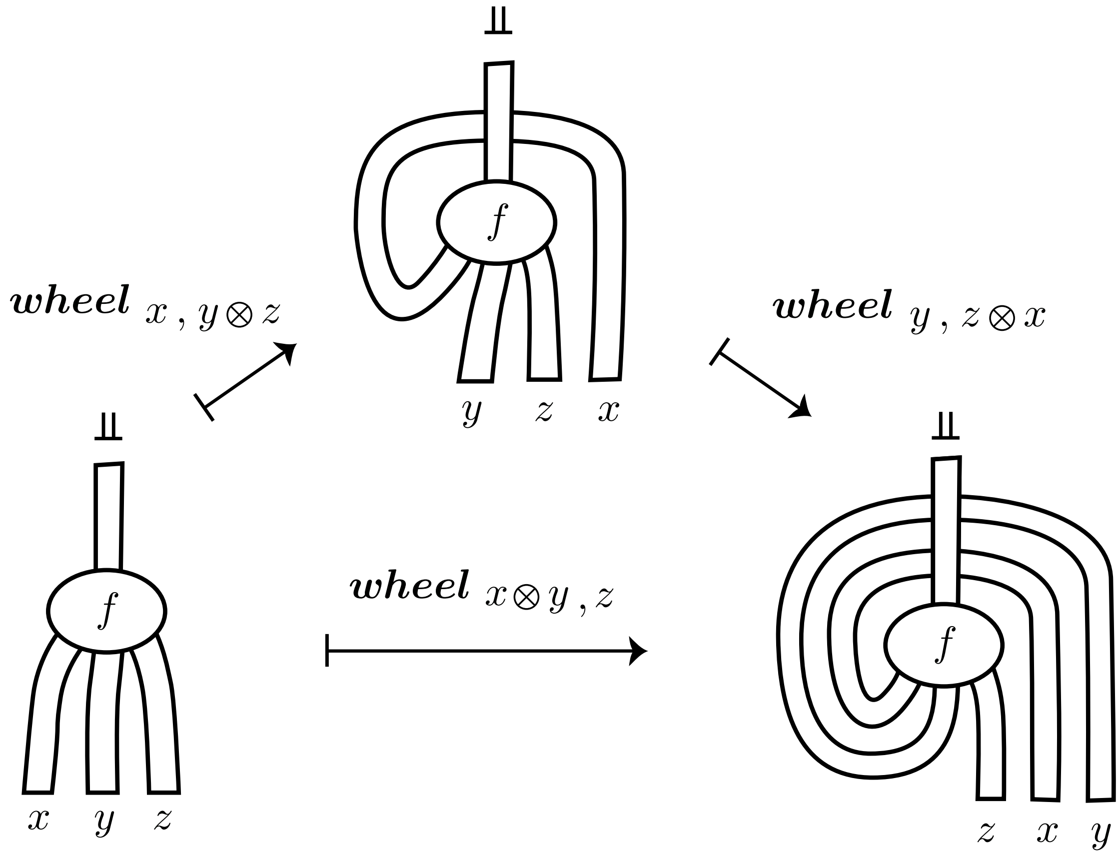
$wheel_{x,y}$:



$\mapsto$



The coherence diagram

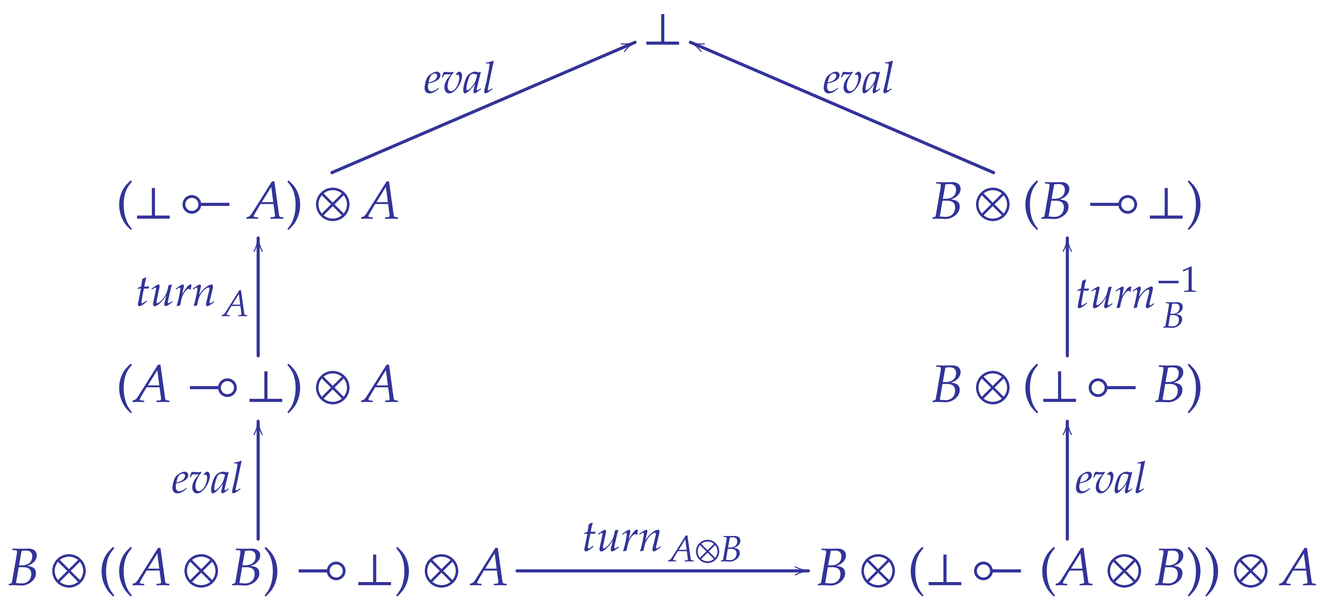


An equivalent formulation

A dialogue category equipped with a natural isomorphism

$$turn_A \quad : \quad A \multimap \bot \quad \longrightarrow \quad \bot \circ\!\!- A$$

making the diagram below commute:

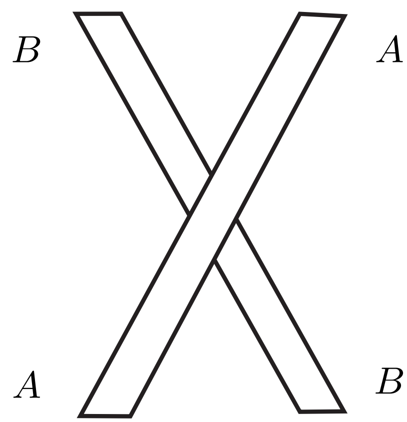


Braided categories

A monoidal category $\mathscr{C}$ equipped with a family of isomorphisms

$$\gamma_{A,B} \quad : \quad A \otimes B \quad \longrightarrow \quad B \otimes A$$

natural in A and B, represented pictorially as the positive braiding

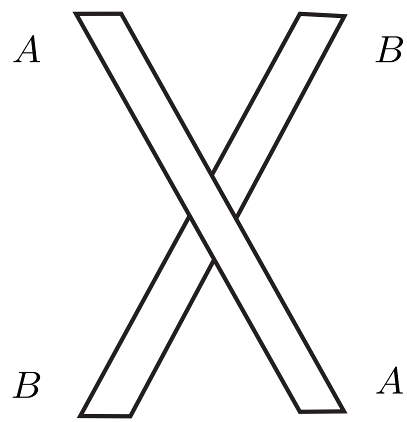


Braided categories

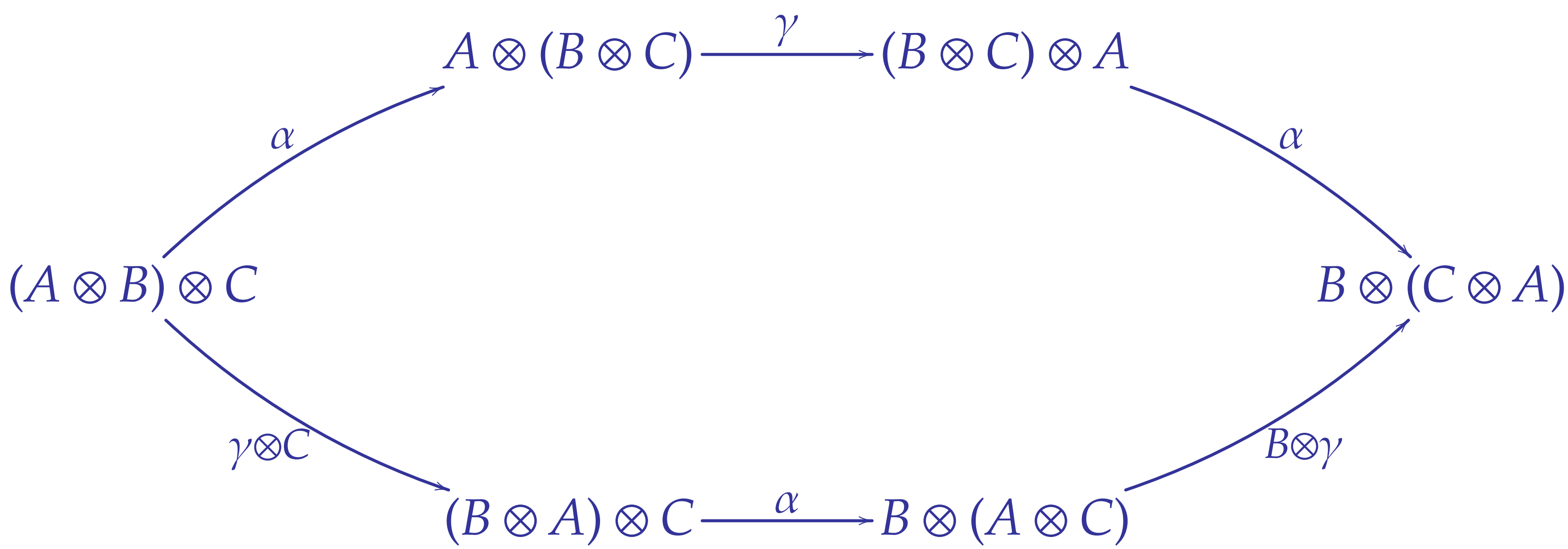
As expected, the inverse map

$$\gamma_{A,B}^{-1} \quad : \quad B \otimes A \quad \longrightarrow \quad A \otimes B$$

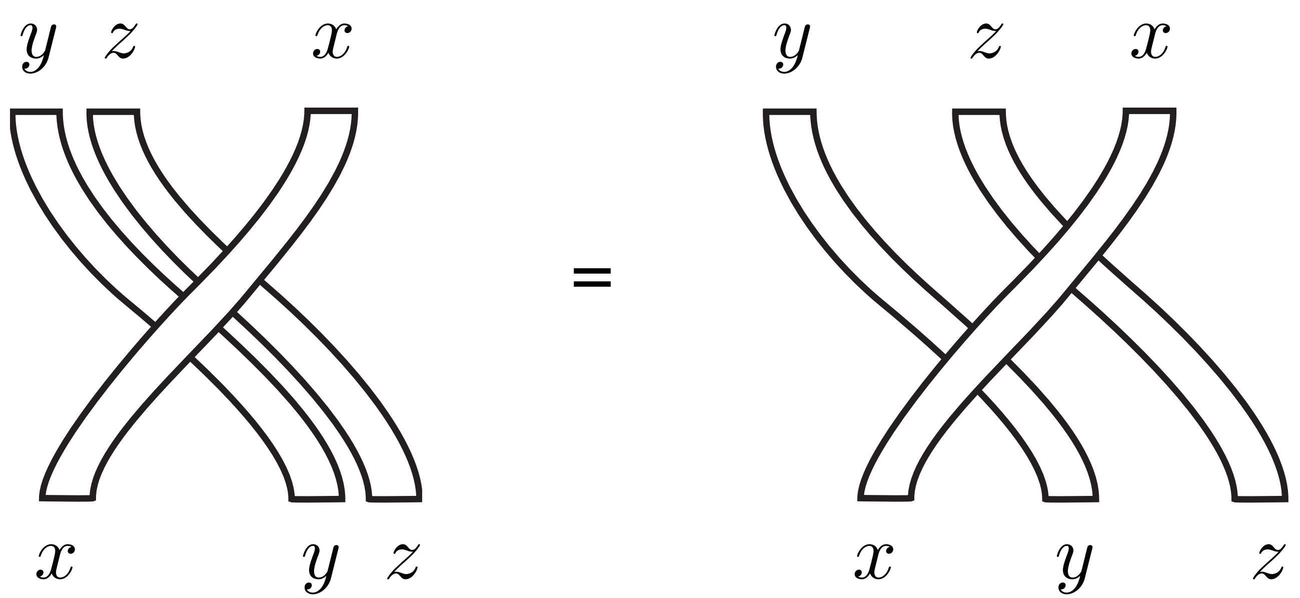
is represented pictorially as the negative braiding



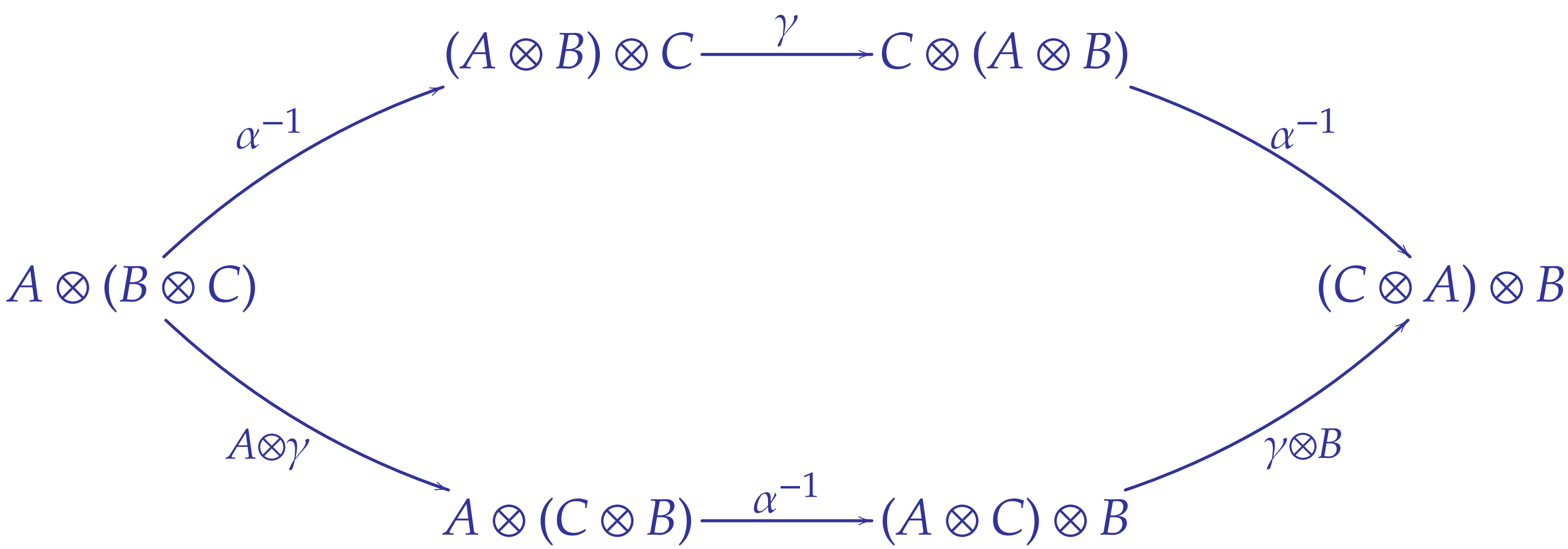
Coherence diagram for braids [1]



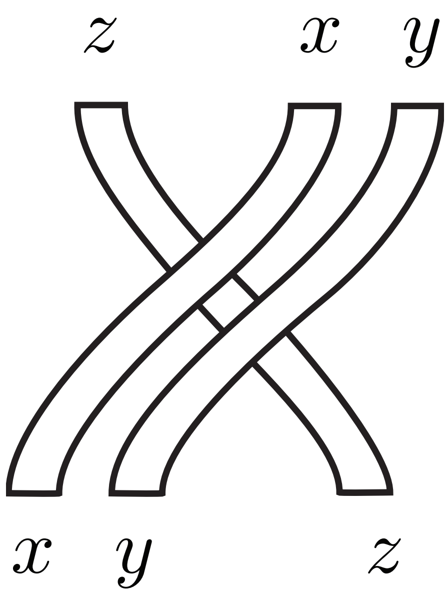
Same coherence diagram in string diagrams



Coherence diagram for braids [2]



Same coherence diagram in string diagrams



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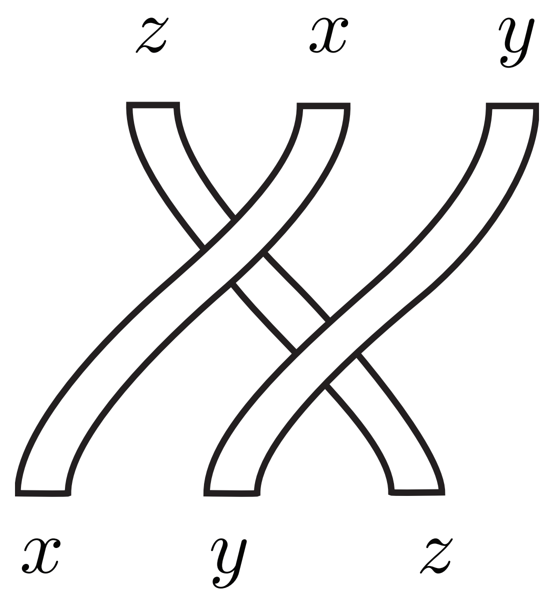
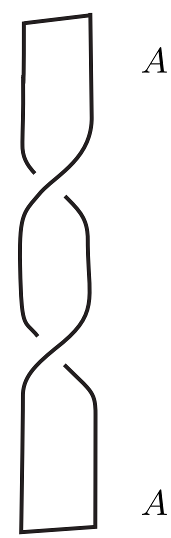


Balanced categories

A braided monoidal category $\mathscr{C}$ equipped with a **twist**

$$\theta_A \quad : \quad A \quad \longrightarrow \quad A$$

defined as a natural family of isomorphisms, and depicted as



Coherence for twists

The twist θ is required to satisfy the equality

$$\theta_I \quad = \quad id_I$$

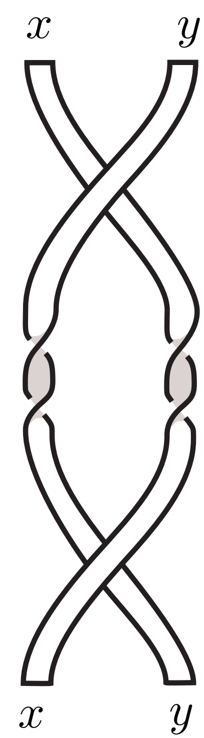
and to make the diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\gamma_{A,B}} & B \otimes A \\ \downarrow{\scriptstyle \theta_{A\otimes B}} & & \downarrow{\scriptstyle \theta_B \otimes \theta_A} \\ A \otimes B & \xleftarrow[\gamma_{B,A}]{} & B \otimes A \end{array}$$

commute for all objects A and B.

Coherence for twists

$\theta_{x\otimes y}$ =



The free balanced dialogue category

The objects of the category **free-dialogue**($\mathscr{C}$) are the **formulas** of tensorial ribbon logic:

$$A, B \quad ::= \quad X \quad | \quad A \otimes B \quad | \quad A \multimap \bot \quad | \quad \bot \multimapinv A \quad | \quad 1$$

where X is an object of the category $\mathscr{C}$.

The morphisms are the **proofs** of the logic modulo equality.

In the case of the free balanced dialogue category

Ribbon logic

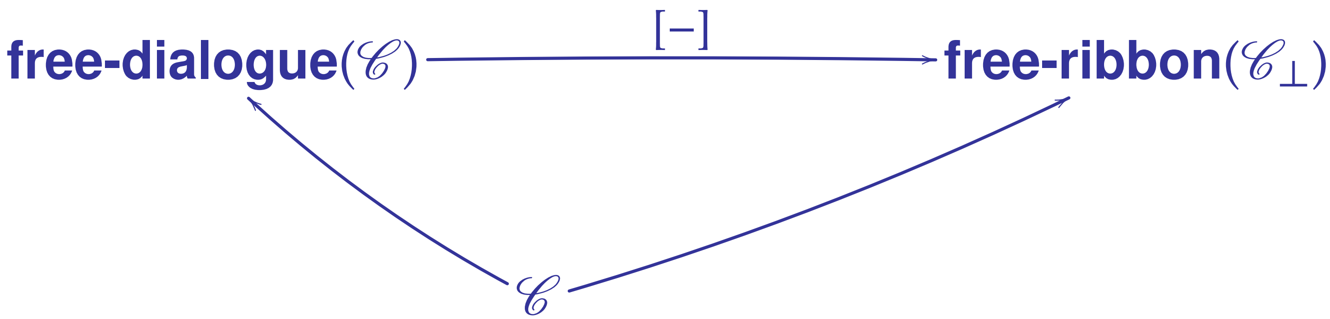
Main ingredient: the exchange rule

$$\text{Exchange } [g] \qquad \frac{A_1, \ldots, A_n \vdash B}{A_{g*1}, \ldots, A_{g*n} \vdash B}$$

is parametrized by the elements $g \in G_n$ of the ribbon group.

A proof-as-tangle theorem

Every category $\mathscr{C}$ of atomic formulas induces a functor $[-]$ such that



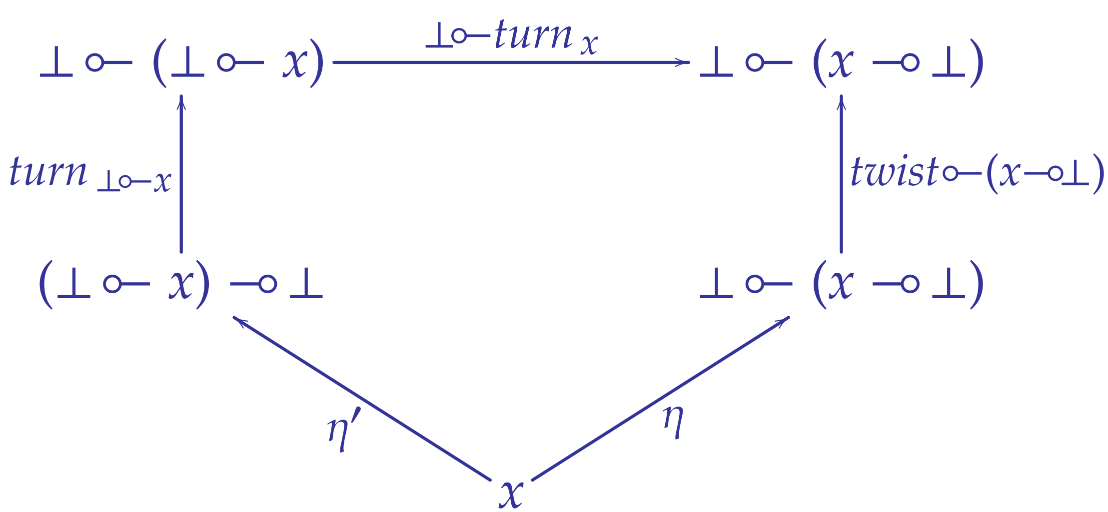
where $\mathscr{C}_\perp$ is the category $\mathscr{C}$ extended with an object $\perp$.

Theorem. The functor $[-]$ is faithful.

$\longrightarrow$ a topological foundation for game semantics

An illustration

Imagine that we want to check that the diagram



commutes in every balanced dialogue category.

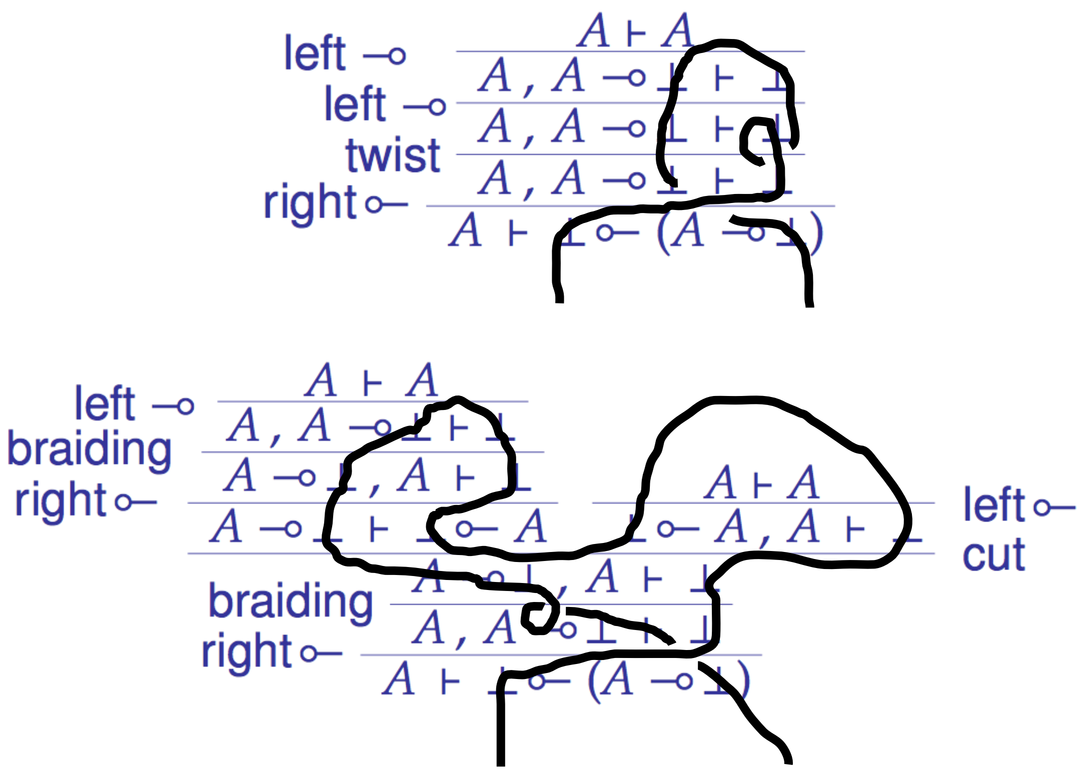
An illustration

Equivalently, we want to check that the two derivation trees below are equal:

$$
\text{right}\multimapinv\;
\cfrac{
  \text{twist}\;
  \cfrac{
    \text{left}\multimap\;
    \cfrac{
      \text{left}\multimap\;
      \cfrac{A \vdash A}{A\,,\,A \multimap \bot \;\vdash\; \bot}
    }{A\,,\,A \multimap \bot \;\vdash\; \bot}
  }{A\,,\,A \multimap \bot \;\vdash\; \bot}
}{A \;\vdash\; \bot \multimapinv (A \multimap \bot)}
$$

$$
\text{right}\multimapinv\;
\cfrac{
  \text{braiding}\;
  \cfrac{
    \cfrac{
      \text{right}\multimapinv\;
      \cfrac{
        \text{braiding}\;
        \cfrac{
          \text{left}\multimap\;
          \cfrac{A \;\vdash\; A}{A\,,\,A \multimap \bot \vdash \bot}
        }{A \multimap \bot\,,\,A \;\vdash\; \bot}
      }{A \multimap \bot \;\vdash\; \bot \multimapinv A}
      \qquad
      \cfrac{A \vdash A}{\bot \multimapinv A\,,\,A \;\vdash\; \bot}\;\text{left}\multimapinv
    }{A \multimap \bot\,,\,A \;\vdash\; \bot}\;\text{cut}
  }{A\,,\,A \multimap \bot \;\vdash\; \bot}
}{A \;\vdash\; \bot \multimapinv (A \multimap \bot)}
$$

An illustration



equality of proofs $\Longleftrightarrow$ equality of tangles

Game semantics in string diagrams

The connection to Guy's tutorial on dialogue games

Main theorem

The objects of the free **symmetric** dialogue category are **dialogue games** constructed by the grammar

$$A, B \quad ::= \quad X \quad | \quad A \otimes B \quad | \quad \neg A \quad | \quad 1$$

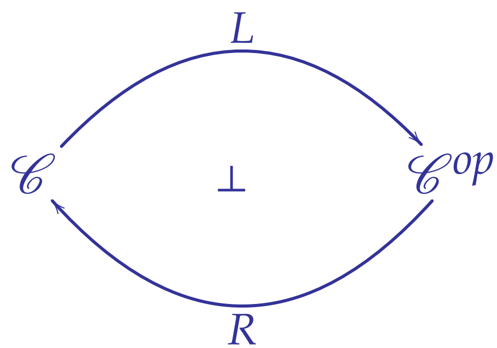
where X is an object of the category $\mathscr{C}$.

The morphisms are **total** and **innocent strategies** on dialogue games.

As we will see: proofs become 3-dimensional variants of knots...

An algebraic presentation of dialogue categories

Negation defines a pair of **adjoint functors**



witnessed by the series of bijection:

$$\mathscr{C}(A, \neg B) \quad \cong \quad \mathscr{C}(B, \neg A) \quad \cong \quad \mathscr{C}^{op}(\neg A, B)$$

An algebraic presentation of dialogue chiralities

The algebraic presentation starts by the pair of **adjoint functors**

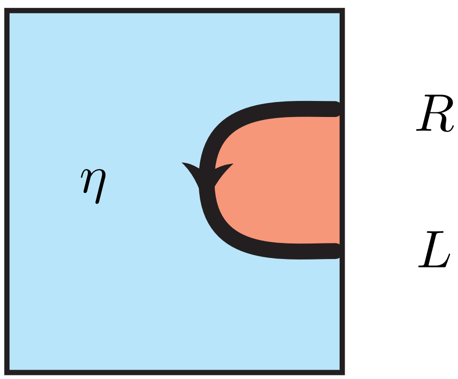
$$\mathscr{A} \underset{R}{\overset{L}{\underset{\bot}{\rightleftarrows}}} \mathscr{B}$$

between the two components $\mathscr{A}$ and $\mathscr{B}$ of the dialogue chirality.

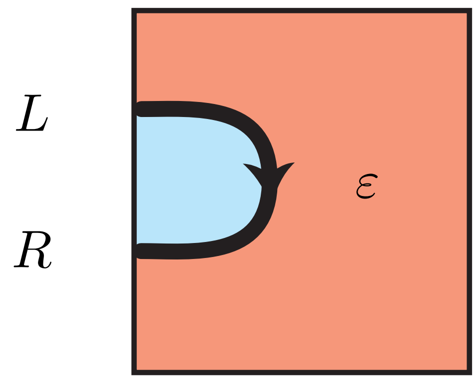
The 2-dimensional topology of adjunctions

The **unit** and **counit** of the adjunction $L \dashv R$ are depicted as

$\eta : Id \longrightarrow R \circ L$



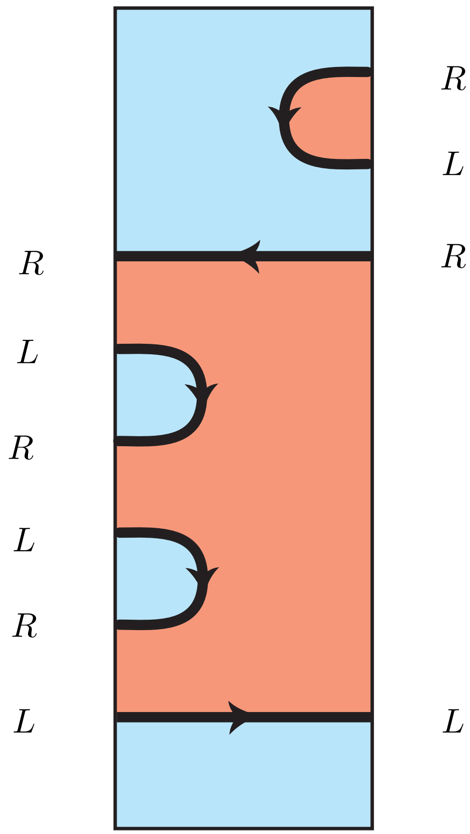
$\varepsilon : L \circ R \longrightarrow Id$



Opponent move = functor R

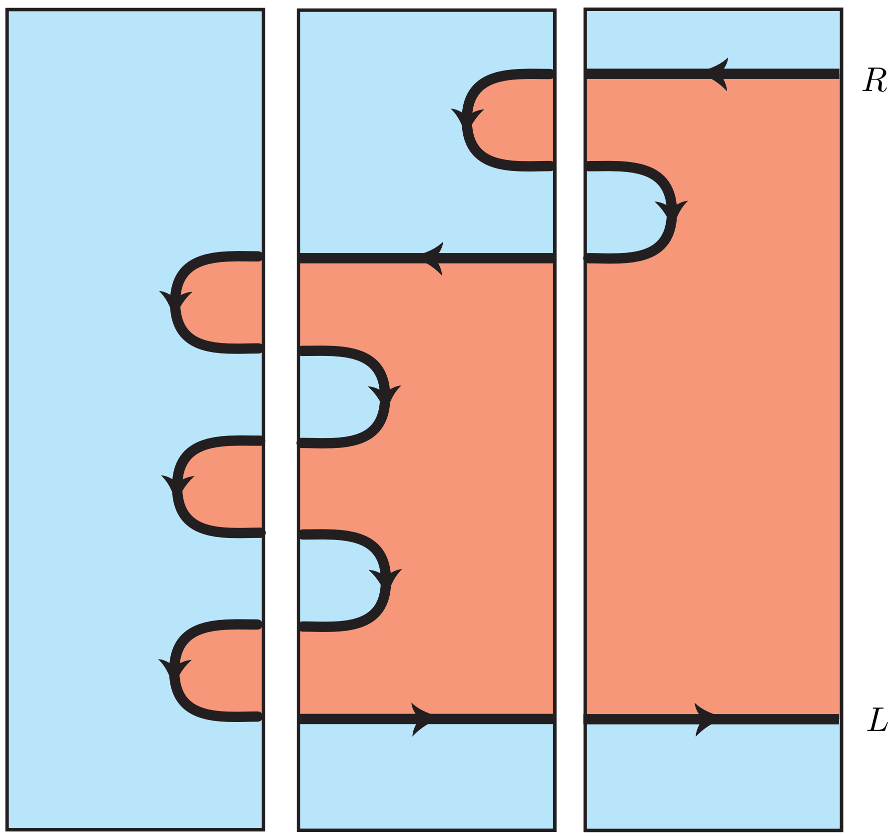
Proponent move = functor L

A typical proof

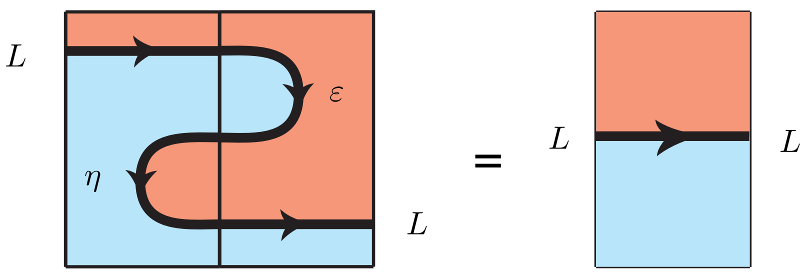


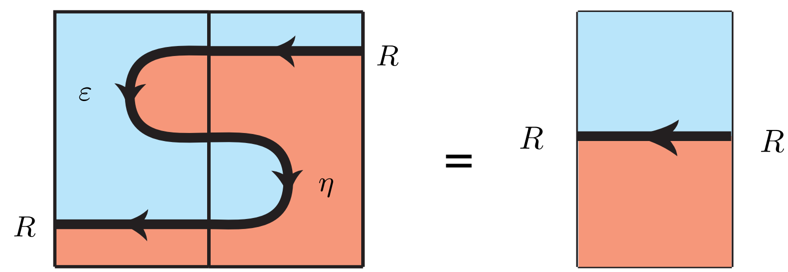
Reveals the algebraic nature of game semantics

A purely diagrammatic cut elimination



The 2-dimensional dynamics of adjunctions





Recovers the usual way to compose strategies in game semantics

When a tensor meets a negation...

The continuation monad is strong

$$(\neg\neg A) \otimes B \longrightarrow \neg\neg(A \otimes B)$$

As Gordon explained, this is the starting point of algebraic effects

Tensor vs. negation

Proofs are generated by a **parametric strength**

$$\kappa_X \quad : \quad \neg\,(X \otimes \neg\, A) \otimes B \longrightarrow \neg\,(X \otimes \neg\,(A \otimes B))$$

which generalizes the usual notion of **strong monad** :

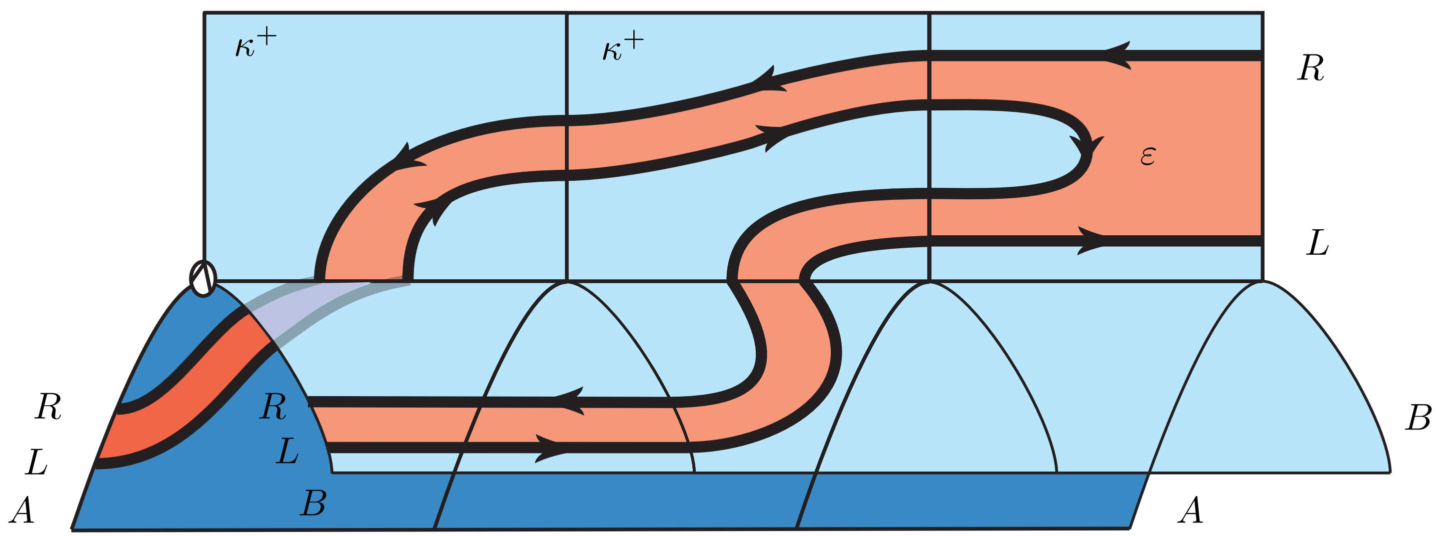
$$\kappa \quad : \quad \neg\neg\, A \otimes B \longrightarrow \neg\neg\,(A \otimes B)$$

Proofs as 3-dimensional string diagrams

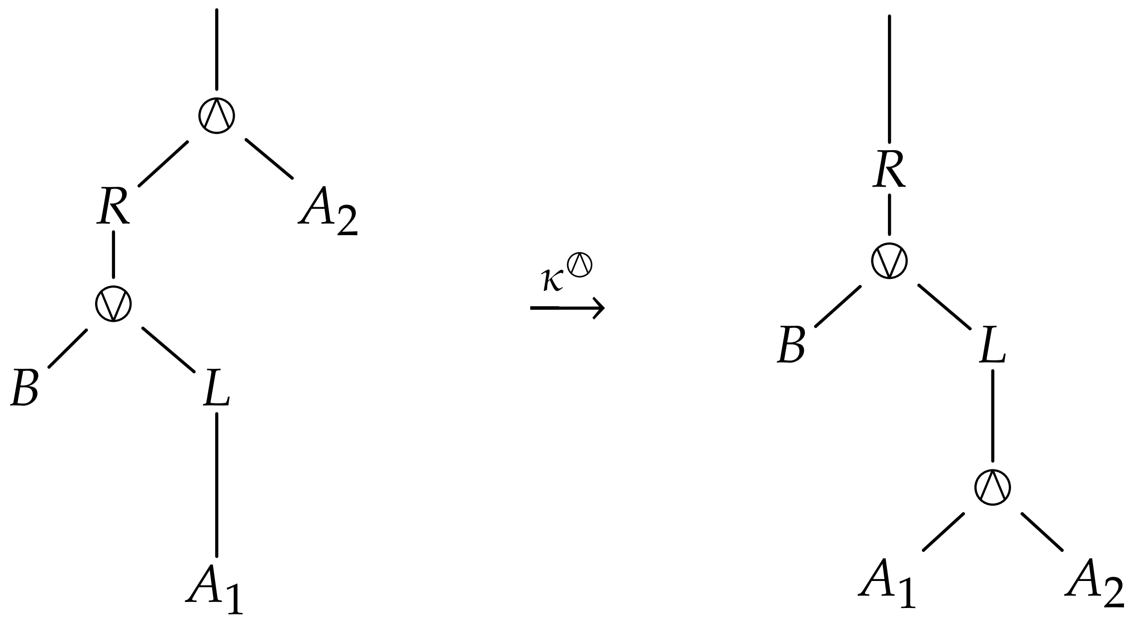
The left-to-right proof of the sequent

$$\neg\neg A \otimes \neg\neg B \quad \vdash \quad \neg\neg(A \otimes B)$$

is depicted as

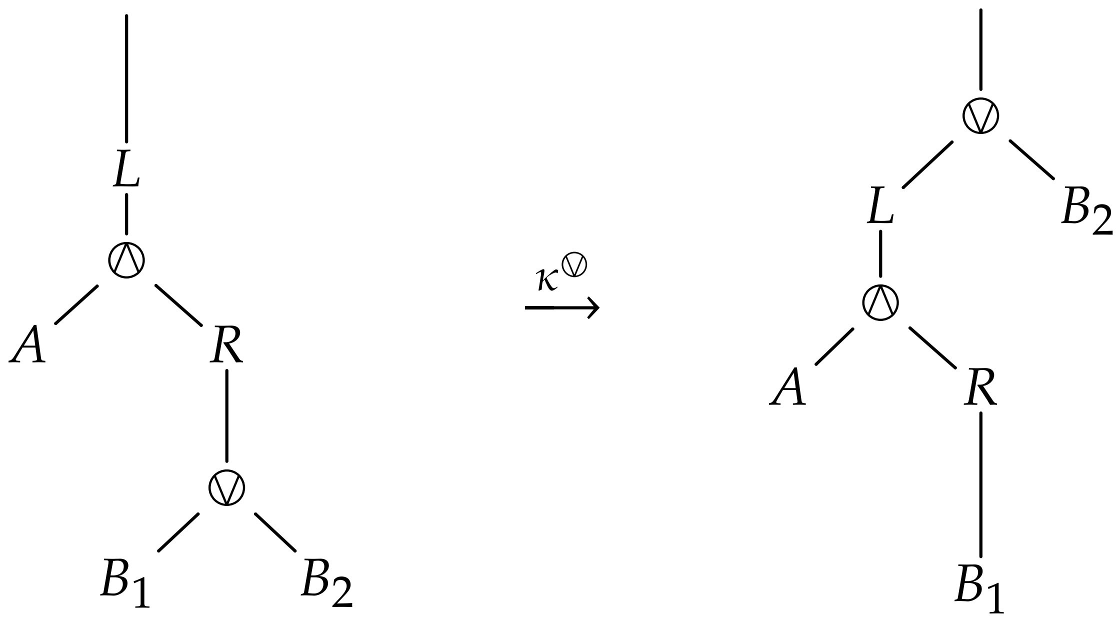


Tensor vs. negation : conjunctive strength



Linear distributivity in a continuation framework

Tensor vs. negation : disjunctive strength



Linear distributivity in a continuation framework

A factorization theorem

The four proofs $\eta, \epsilon, \kappa^{\owedge}$ and $\kappa^{\ovee}$ generate every proof of the logic. Moreover, every such proof

$$X \xrightarrow{\epsilon} \xrightarrow{\kappa^{\owedge}} \xrightarrow{\epsilon} \xrightarrow{\epsilon} \xrightarrow{\eta} \xrightarrow{\eta} \xrightarrow{\kappa^{\ovee}} \xrightarrow{\epsilon} \xrightarrow{\eta} \xrightarrow{\epsilon} \xrightarrow{\kappa^{\ovee}} \xrightarrow{\eta} \xrightarrow{\eta} Z$$

factors uniquely as

$$X \overset{\kappa^{\owedge}}{\twoheadrightarrow} \overset{\epsilon}{\twoheadrightarrow} \overset{\eta}{\twoheadrightarrow} \overset{\kappa^{\ovee}}{\twoheadrightarrow} Z$$

This factorization reflects a Player – Opponent view factorization

Axiom and cut links

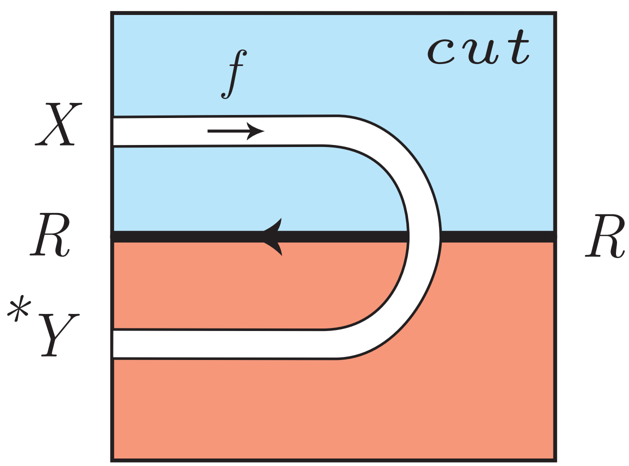
The basic building blocks of linear logic

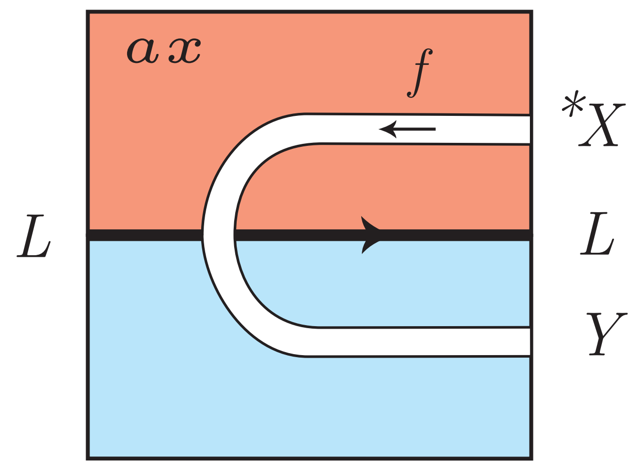
Axiom and cut links

Every map

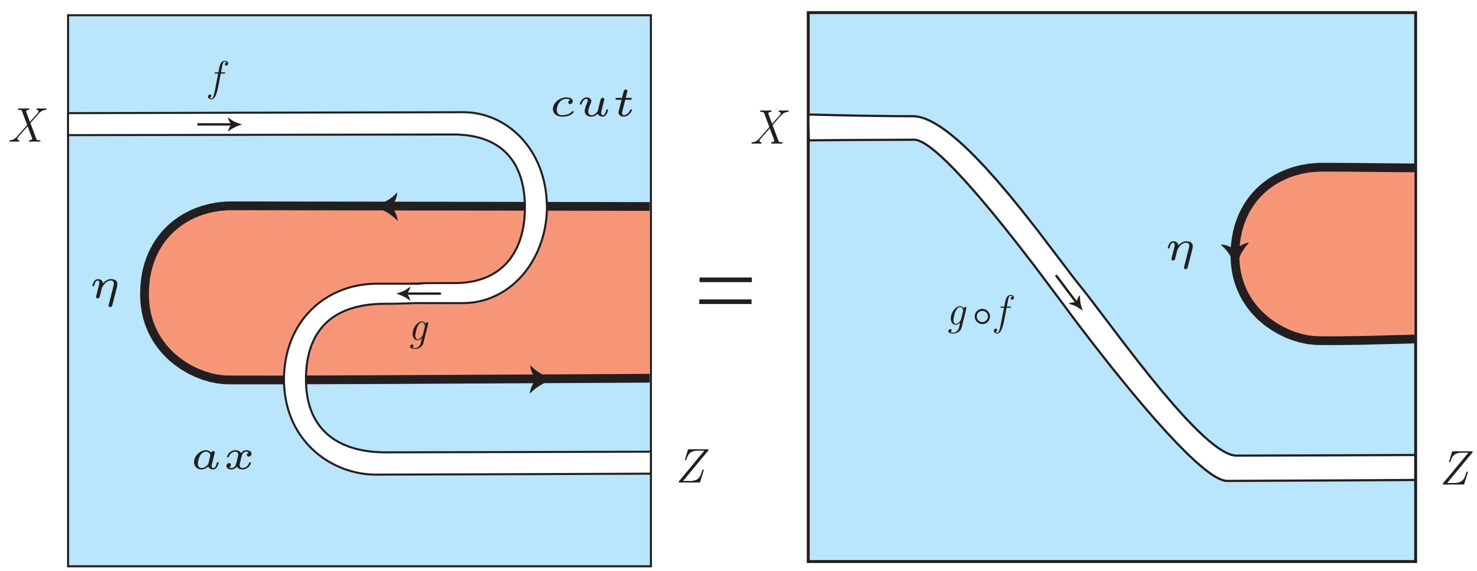
$$f \quad : \quad X \quad \longrightarrow \quad Y$$

between atoms in the category $\mathscr{C}$ induces an axiom and a cut combinator:

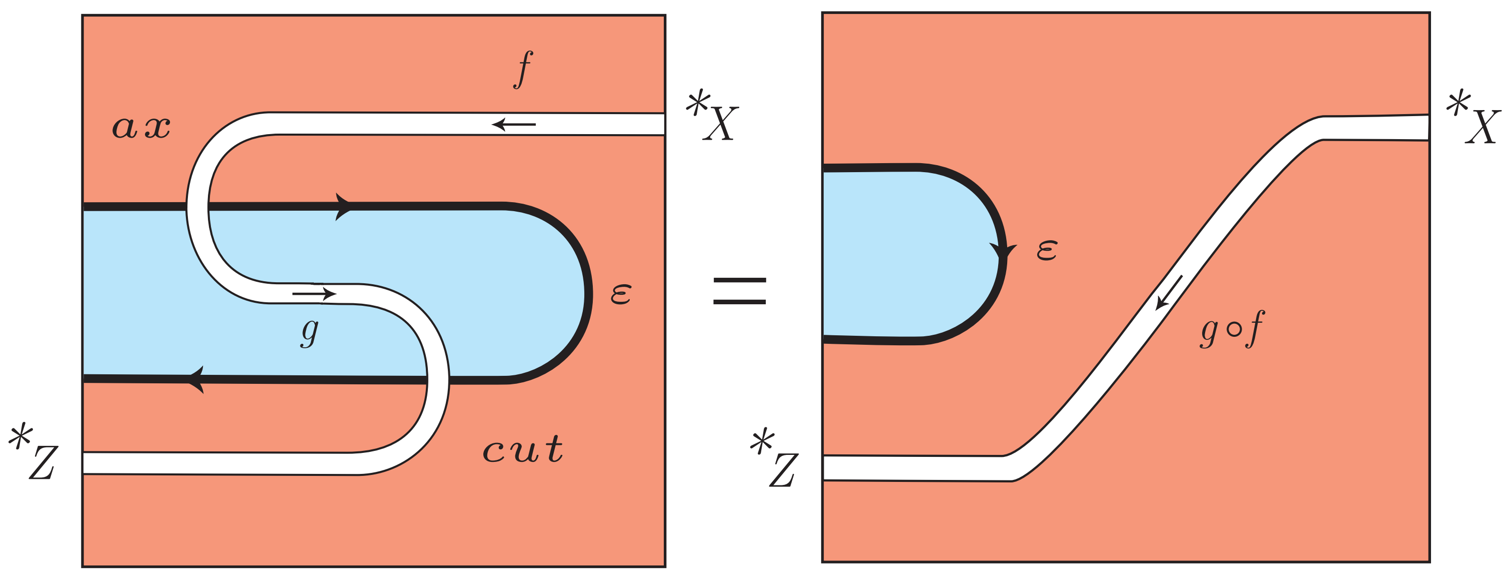




Equalities between axiom and cut links



Equalities between axiom and cut links



Local stores

A diagrammatic account of the local state monad

Algebraic presentations of effects

We want to reason about **programs with effects** like states, exceptions...

Computational monads:

$$A \xrightarrow{\text{impure}} B \quad = \quad A \xrightarrow{\text{pure}} T(B)$$

Equational theories:

$$\text{operations} \quad : \quad A^n \longrightarrow A \qquad \text{and} \qquad \text{equations}$$

Presheaf models

Key idea: interpret a type A as a family of sets

$$A_{[0]} \qquad A_{[1]} \qquad \cdots \qquad A_{[n]} \qquad \cdots$$

indexed by natural numbers, where each set

$$A_{[n]}$$

contains the programs of type A which have access to n variables.

Presheaf models

This defines a covariant presheaf

$$A_{[n]} \quad : \quad Inj \quad \longrightarrow \quad Set$$

on the category Inj of natural numbers and injections.

The action of the injections on A are induced by the operations

$$\mathtt{dispose}_{\langle loc \rangle} \quad : \quad A_{[n]} \quad \longrightarrow \quad A_{[n+1]}$$

defined for $0 \leq loc \leq n$.

Local stores [Plotkin & Power 2002]

The slightly intimidating monad

$$TA \quad : \quad n \quad \mapsto \quad S^{[n]} \quad \Rightarrow \quad \left(\quad \int^{p \in Inj} \quad S^{[p]} \times A_{[p]} \times I(n, p) \quad \right)$$

on the presheaf category $[Inj, Set]$ where the contravariant presheaf

$$S^{[p]} \quad = \quad V^{p}$$

describes the states available at degree p.

Key theorem [Plotkin & Power 2002]

the category of mnemoids

is equivalent to

the category of algebras of the state monad

This provides an algebraic presentation of the state monad

Mnemoids

A **mnemoid** is a family of sets

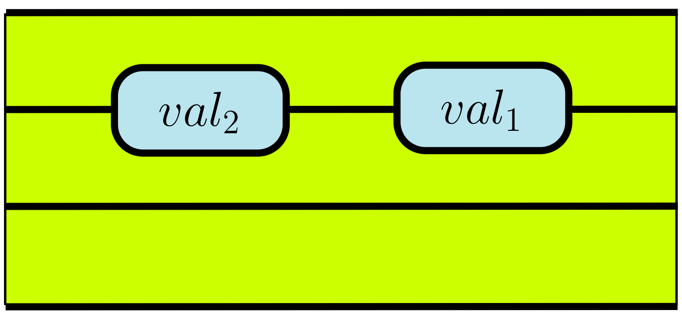
$$A_{[0]} \qquad A_{[1]} \qquad \cdots \qquad A_{[n]} \qquad \cdots$$

equipped with the following operations

$$\begin{array}{lcccc}
\texttt{lookup}_{\langle loc \rangle} & : & A_{[n]}^{V} & \longrightarrow & A_{[n]} \\
\texttt{update}_{\langle loc,val \rangle} & : & A_{[n]} & \longrightarrow & A_{[n]} \\
\texttt{fresh}_{\langle loc,val \rangle} & : & A_{[n+1]} & \longrightarrow & A_{[n]} \\
\texttt{dispose}_{\langle loc \rangle} & : & A_{[n]} & \longrightarrow & A_{[n+1]}
\end{array}$$

satisfying a series of basic equations.

Interaction update – update



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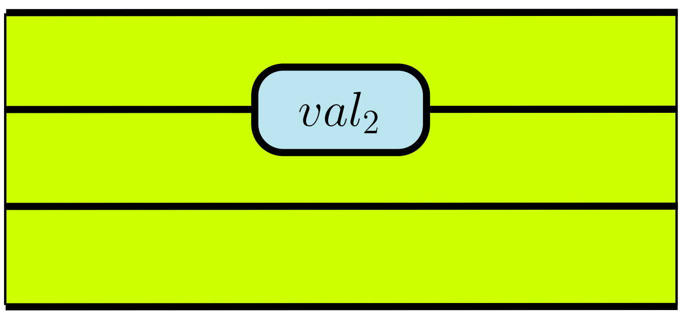
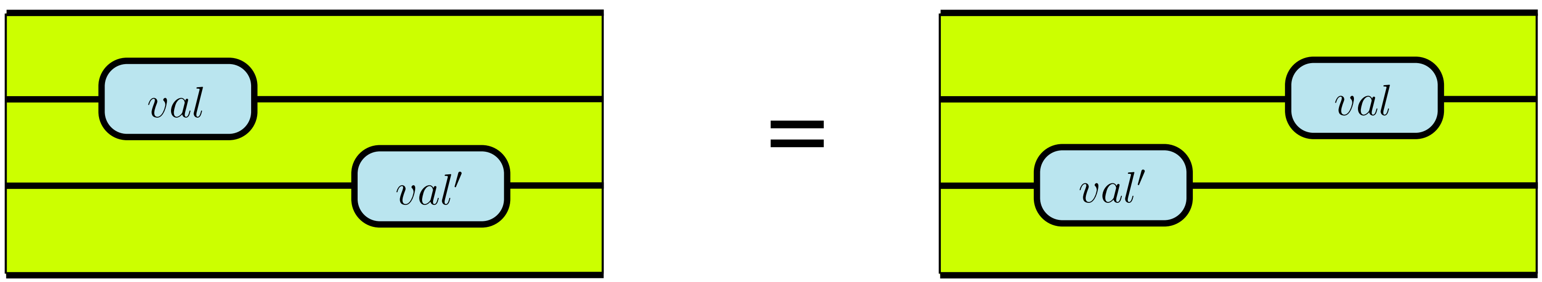
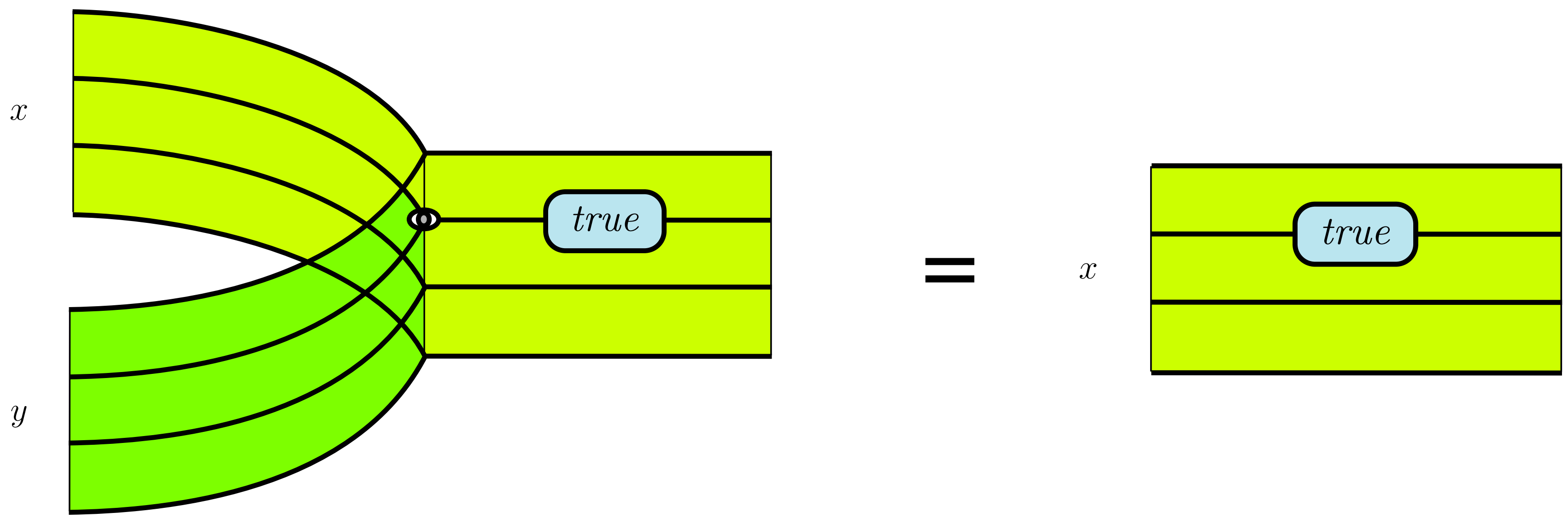


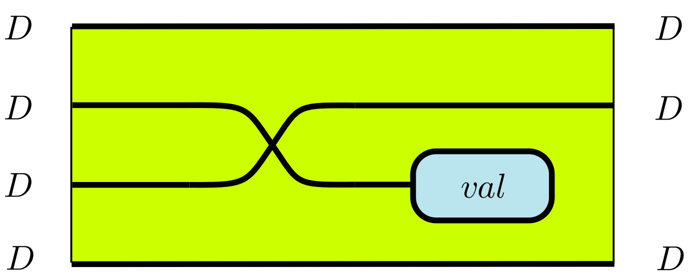
Commutation update – update



Interaction update – lookup



Interaction fresh – permutation



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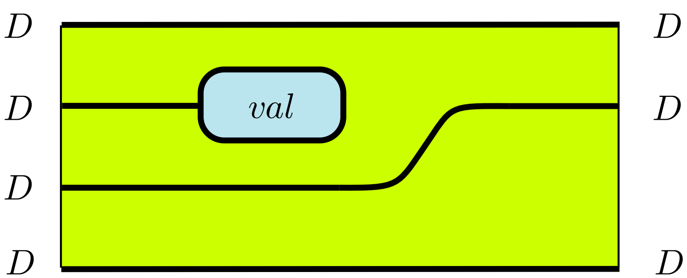
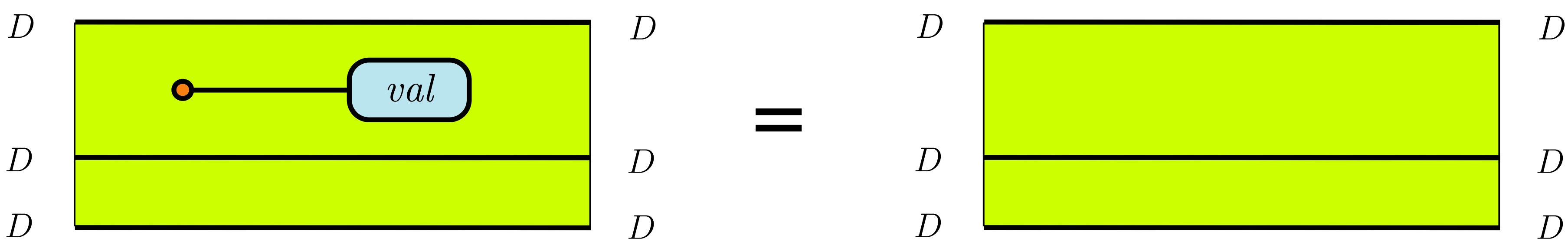
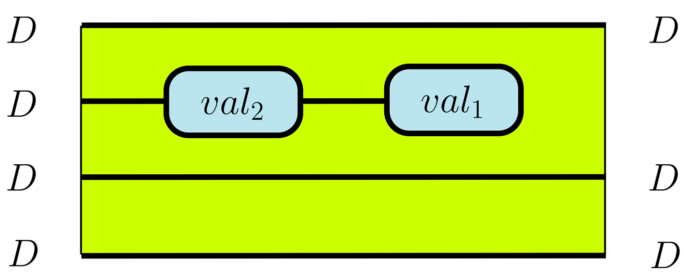


Garbage collect : fresh – dispose



Interaction fresh – update



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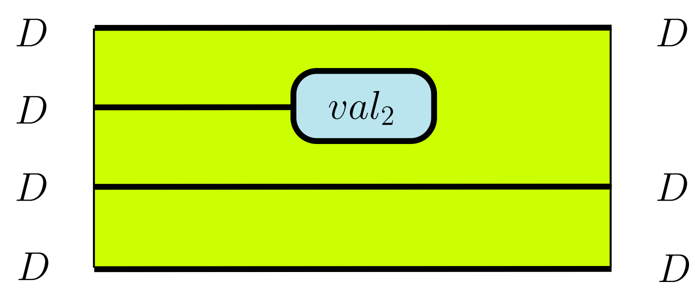
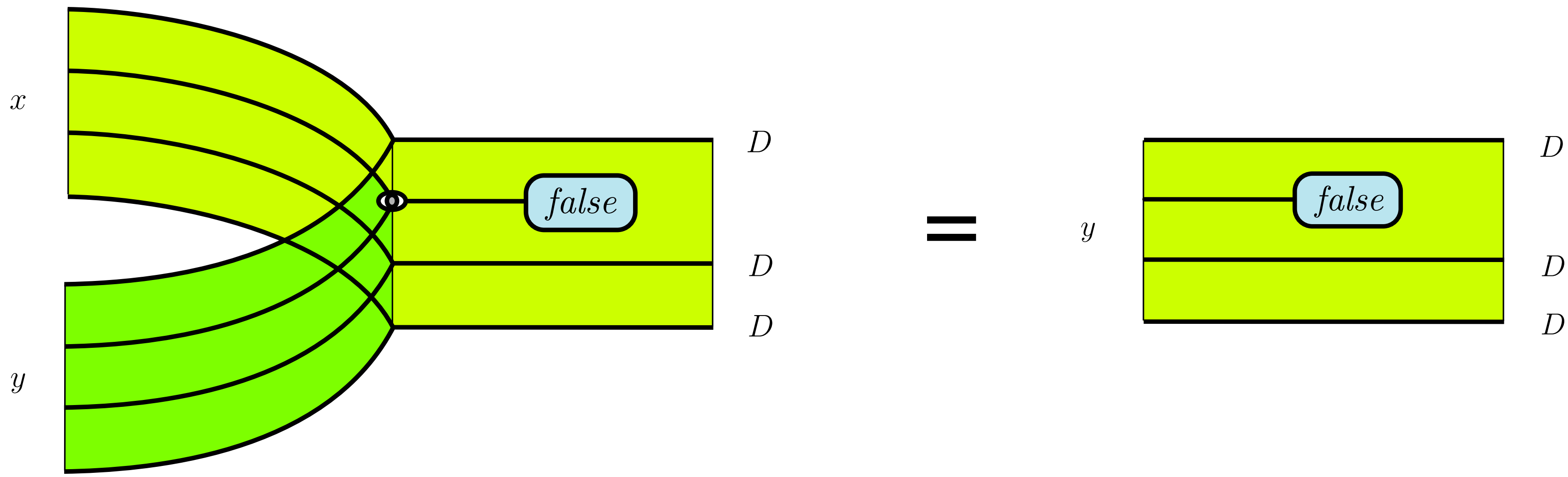


Interaction fresh – lookup



Tensorial logic with local stores

Beware: work in progress !

A dialogue category with local stores

A family of return types

$$\bot_{[0]} \qquad \bot_{[1]} \qquad \cdots \qquad \bot_{[n]} \qquad \cdots$$

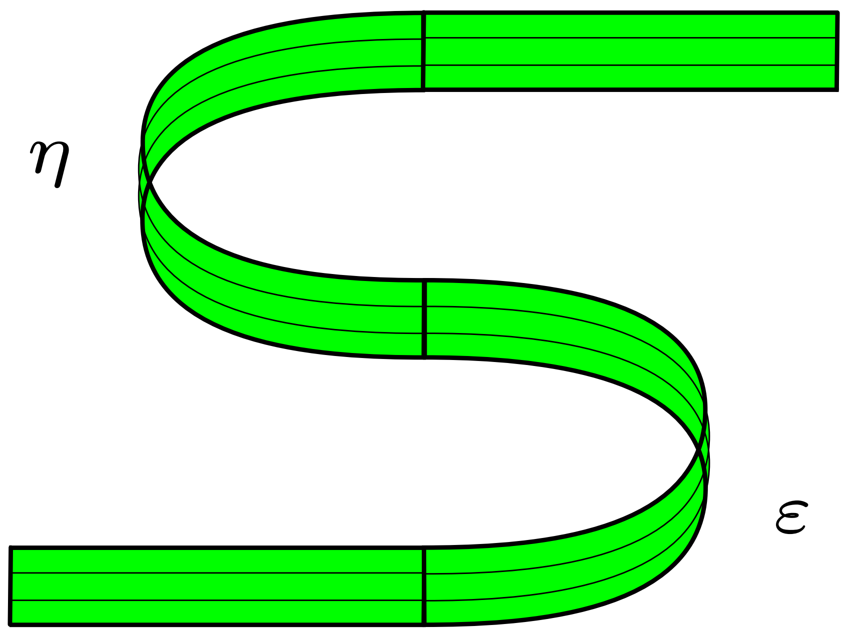
equipped with the following operations

$$\begin{array}{lcccc}
\mathtt{lookup}_{\langle loc \rangle} & : & \bot_{[n]}^{V} & \longrightarrow & \bot_{[n]} \\
\mathtt{update}_{\langle loc,val \rangle} & : & \bot_{[n]} & \longrightarrow & \bot_{[n]} \\
\mathtt{fresh}_{\langle loc,val \rangle} & : & \bot_{[n+1]} & \longrightarrow & \bot_{[n]} \\
\mathtt{dispose}_{\langle loc \rangle} & : & \bot_{[n]} & \longrightarrow & \bot_{[n+1]}
\end{array}$$

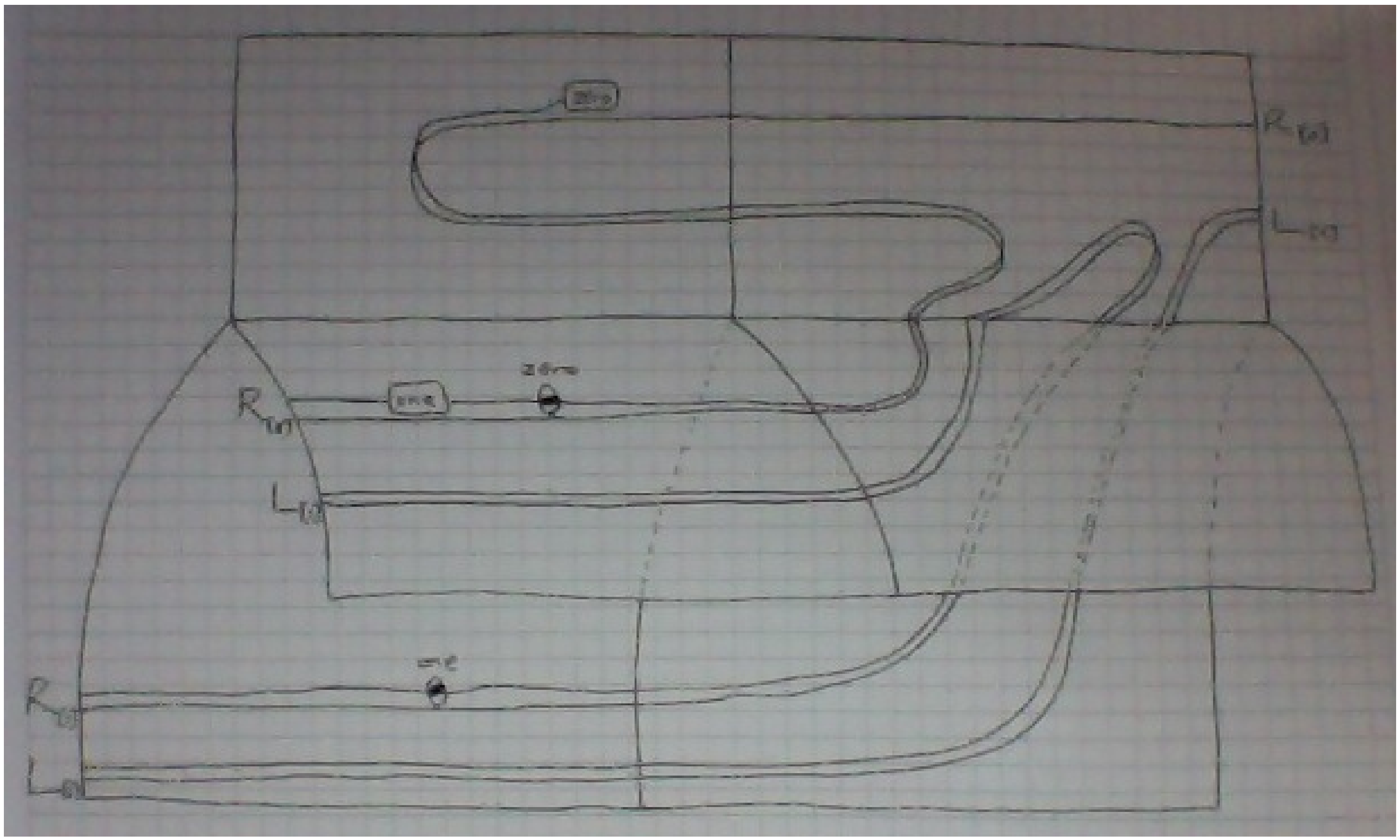
satisfying the equations of a mnemoid.

Game semantics with local stores

Graphically:



The prototype of a visible (non innocent) strategy



Game semantics with local stores

Fact. There is a nice and interesting definition of

the free dialogue category $\mathscr{M}$ with a mnemoid pole

formulated in the language of game semantics.

Observation: there is a canonical functor

$$\mathscr{M} \quad \longrightarrow \quad [Inj, Set]$$

obtained by taking

$$\bot_{[n]} \quad : \quad p \quad \mapsto \quad T(A)(n+p)$$

for any presheaf A in the category $[Inj, Set]$. Typically, take $A = 1$.

Work in progress

Devise a **neat** categorical definition of

a dialogue category with local ground stores

such that the free such dialogue category coincides with the category

– with arena games as objects,

– with visible strategies as morphisms.

I have a definition at this point, but not yet entirely satisfactory...