The Classical Realizability Tripos and Topos

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Categorical Approach to Realizability

Following an idea of D. Scott around 1980 M. Hyland introduced the **effective topos** and more generally **realizability toposes** induced by **partial combinatory algebras** (pca's).

Previously known were **boolean** and **Heyting** valued models.
As a unifying concept Hyland, Johnstone and Pitts in 1980 introduced the notion of **tripos**. With every tripos one may associate a topos by *adding quotients*.

In this talk we survey the basics of tripos theory and will show that Krivine's classical realizability gives rise to a tripos and ensuing topos. We formulate and discuss some basic questions about its structure.

Tripos (1)

A **tripos** (over $\mathbf{Set}$) is a (pseudo-)functor $\mathsf{P} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{pHeyt}$ satisfying some properties.

The elements of $\mathsf{P}(I)$ are thought of as *predicates on* I and the preorder relation $\vdash_I$ is thought of as *entailment*.

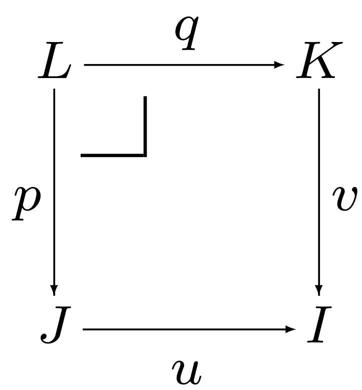
For $u : J \to I$ the logic preserving map $\mathsf{P}(u) : \mathsf{P}(I) \to \mathsf{P}(J)$ is denoted by u^* and thought of as substitution.

Tripos (2)

Following Lawvere **quantifiers** are given as **adjoints to substitution**

(1) for all $u : J \to I$ we have $\exists_u \dashv u^* \dashv \forall_u$

(2) $u^*\exists_v \dashv\vdash \exists_p q^*$ and $u^*\forall_v \dashv\vdash \forall_p q^*$ for every pullback



where (2) states how quantification commutes with substitution.

Tripos (3)

There is a set Σ of "propositions" and a "truth predicate" $\mathsf{Tr} \in \mathsf{P}(\Sigma)$ which is **generic** in the sense that

for all $\varphi \in \mathsf{P}(I)$ we have $\varphi \dashv\vdash f^*\mathsf{Tr}$ for some $f : I \to \Sigma$

NB typically f is not unique!

Σ^I is the set of predicates on I. Reindexing Tr along the evaluation map $I \times \Sigma^I$ gives rise to the predicate $\in_I$.

NB One can interpret **higher order logic** in a tripos. The generic predicate Tr allows one to interpret the **comprehension axiom** but in general **extensionality** does not hold.

Examples of Triposes

(1) For a complete Heyting algebra Ω one may consider the tripos where $\mathsf{P}(I)$ is Ω^I ordered pointwise. Quantification is given by (infinite) meets and joins. We have $\Sigma = \Omega$ and Tr is given by id_Ω.

(2) Let $\mathbb{A}$ be a **partial combinatory algebra** (pca), e.g. $\mathbb{N}$ with Kleene application $\{\cdot\}(\cdot)$ or a model of untyped λ-calculus. One may consider the tripos where $\mathsf{P}(I) = \mathcal{P}(\mathbb{A})^I$ and $\varphi \vdash_I \psi$ iff

$$\exists a \in \mathbb{A}. \forall i \in I. \forall b \in \varphi_i.\ ab \in \psi_i$$

We have $\Sigma = \mathcal{P}(\mathbb{A})$ and Tr is given by $\mathsf{id}_{\mathcal{P}(\mathbb{A})}$.

Tripos to Topos Construction

For a tripos P one may construct its associated topos $\mathbf{Set}[\mathsf{P}]$. Its objects are pairs $X = (|X|, E_X)$ where $|X|$ is a set and $E_X \in \mathsf{P}(|X|{\times}|X|)$ is symmetric and transitive in the sense of the logic of P.
We write $E_X(x)$ for $E_X(x,x)$.

A morphism from X to Y is a relation $F \in \mathsf{P}(|X|{\times}|Y|)$ which is **functional** in the sense that

(strict)	$F(x,y) \vdash E_X(x) \wedge E_Y(y)$
(cong)	$E_X(x,x') \wedge F(x,y) \wedge E_Y(y,y') \vdash F(x',y')$
(singval)	$F(x,y) \wedge F(x,y') \vdash E_Y(y,y')$
(tot)	$E_X(x) \vdash \exists y{:}|Y|.F(x,y)$

where F is identified with all other such F' which are logically equivalent, i.e. $F(x,y) \dashv\vdash F'(x,y)$.

Equality in Triposes

For every set I we have an **equality** predicate $\mathsf{eq}_I = \exists_{\delta_I} \top_I \in \mathsf{P}(I{\times}I)$ satisfying

$$\top_I \vdash_I \delta_I^* \varrho \qquad \text{iff} \qquad \mathsf{eq}_I \vdash_{I\times I} \varrho$$

for all $\varrho \in \mathsf{P}(I{\times}I)$.
Notice that $\mathsf{eq}_I(i,j)$ is logically equivalent to

$$\forall p{:}\Sigma^I.p(i) \to p(j)$$

i.e. Leibniz equality.

We have a functor $\Delta : \mathbf{Set} \to \mathbf{Set}[\mathsf{P}]$ where $\Delta(I) = (I, \mathsf{eq}_I)$ and $\Delta(u{:}J{\to}I) : \Delta(J) \to \Delta(I)$ is given by the predicate $\mathsf{eq}_I(u(j), i)$.

Properties of Δ and Iteration

The functor $\Delta : \mathbf{Set} \to \mathbf{Set}[\mathsf{P}]$ preserves finite limits and it preserves (regular) epis iff P is **separated**, i.e. reindexing along epis reflects $\vdash$.

Up to equivalence functors of this form can be characterized as finite limit preserving functors F from $\mathbf{Set}$ to a topos $\mathcal{E}$ such that every $A \in \mathcal{E}$ appears as subquotient of some FI. Such an F is regular iff it corresponds to a separated tripos.

Regular such functors are closed under composition which amounts to an **Iteration Theorem** for separated triposes (over toposes).

Almost all triposes are separated in particular all those we will see!

Classical Realizability (1)

The collection of (possibly open) **terms** is given by the grammar

$$t ::= x \mid \lambda x.t \mid ts \mid \mathsf{cc}\, t \mid \mathsf{k}_\pi$$

where π ranges over **stacks**, i.e. expressions of the form $t_1. \cdots .t_n.\pi_0$ where $\pi_0 \in \Pi_0$ is a stack constant and the t_i are closed terms.

A **quasi-proof** is a term without occurrences of k. We write QP for the set of quasi-proofs.

We write Λ for the set of closed terms, QP for the set of closed quasi-proofs and Π for the set of stacks of closed terms.

A **process** is a pair $t * \pi$ with $t \in \Lambda$ and $\pi \in \Pi$.

Classical Realizability (2)

The operational semantics is given by the relation $\succeq$ (*head reduction*) on processes defined inductively by the clauses

(pop)	$\lambda x.t * s.\pi$	$\succeq$	$t[s/x] * \pi$
(push)	$ts * \pi$	$\succeq$	$t * s.\pi$
(store)	$\mathsf{cc}\, t * \pi$	$\succeq$	$t * \mathsf{k}_\pi.\pi$
(restore)	$\mathsf{k}_\pi * t.\pi'$	$\succeq$	$t * \pi$

There is no equality on terms, in particular no β-equality which would be in conflict with certain constants to be added.

Denotational Model of the Language

This language has a natural interpretation within the recursive domain

$$D \cong \Sigma^{\mathsf{List}(D)} \cong \prod_{n \in \omega} \Sigma^{D^n}$$

We have $D \cong \Sigma \times D^D$. Thus D^D is a retract of D and, accordingly, D is a model for λ_β-calculus. The interpretation of Λ is given by

$$\begin{aligned}
&[\![x]\!]_\varrho = \varrho(x) && [\![ts]\!]_\varrho\, k = [\![t]\!]\,\varrho\langle [\![s]\!]_\varrho, k\rangle \\
&[\![\lambda x.t]\!]_\varrho \langle\rangle = \top && [\![\lambda x.t]\!]_\varrho \langle d, k\rangle = [\![t]\!]_{\varrho[d/x]}\, k \\
&[\![\mathsf{cc}\, t]\!]_\varrho\, k = [\![t]\!]_\varrho \langle \mathsf{ret}(k), k\rangle && [\![\mathsf{k}_\pi]\!]_\varrho = \mathsf{ret}([\![\pi]\!]_\varrho)
\end{aligned}$$

where

$$\begin{aligned}
&\mathsf{ret}(k)\langle\rangle = \top && \mathsf{ret}(k)\langle d, k'\rangle = d(k) \\
&[\![\langle\rangle]\!]\,\varrho = \langle\rangle && [\![t.\pi]\!]_\varrho = \langle [\![t]\!]_\varrho, [\![\pi]\!]_\varrho\rangle
\end{aligned}$$

Classical Realizability (3)

A **pole** is a set $\perp\!\!\!\perp$ of processes s.t. $q \in \perp\!\!\!\perp$ whenever $q \succeq p \in \perp\!\!\!\perp$.
We write $t \perp \pi$ for $t * \pi \in \perp\!\!\!\perp$.

For $X \subseteq \Pi$ and $Y \subseteq \Lambda$ we put

$$X^\perp = \{t \in \Lambda \mid \forall \pi \in X.\, t \perp \pi\} \qquad Y^\perp = \{\pi \in \Pi \mid \forall t \in Y.\, t \perp \pi\}$$

Obviously $(-)^\perp$ is antitonic and $Z \subseteq Z^{\perp\perp}$ and thus $Z^\perp = Z^{\perp\perp\perp}$.

If a pole $\perp\!\!\!\perp$ is also closed under head reduction, i.e. $\perp\!\!\!\perp \ni p \succeq q$ implies $q \in \perp\!\!\!\perp$, then $\perp\!\!\!\perp$ corresponds to a subset of $\Lambda * \Pi_0$. If there is just one stack constant the pole corresponds to a subset of Λ thought of as proofs of $\perp$ (falsity).

Classical Realizability (4)

For a pole $\perp\!\!\!\perp$ second order logic over a set M of individuals is interpreted as follows: n-ary predicate variables range over functions $M^n \to \mathcal{P}(\Pi)$ and formulas A are interpreted as $||A|| \subseteq \Pi$ with

$$||X(t_1,\ldots,t_n)||_\varrho = \varrho(X)([\![t_1]\!]_\varrho,\ldots,[\![t_1]\!]_\varrho)$$
$$||A{\to}B||_\varrho = ||A||_\varrho^\perp.||B||_\varrho$$
$$||\forall x A(x)||_\varrho = \bigcup_{a\in M} ||A||_{\varrho[a/x]}$$
$$||\forall X A[X]||_\varrho = \bigcup_{R\in\mathcal{P}(\Pi)^{M^n}} ||A||_{\varrho[R/X]}$$

where ϱ is a valuation of variables.
One may also have constants for functions on M for building terms.

$||A||_\varrho^\perp$ is the **falsity value** and $|A|_\varrho = ||A||_\varrho^\perp$ is the **truth value** of A.

The proposition A is **valid** iff $|A|_\varrho \cap \mathsf{QP} \neq \emptyset$.

Classical Realizability (5)

We have $|\forall X A| = \bigcap_{R \in \mathcal{P}(\Pi)^{M^n}} |A[R/X]|$.

In general $|A \to B|$ is a **proper** subset of

$$|A| \to |B| = \{t \in \Lambda \mid \forall s \in |A| \; ts \in |B|\}$$

unless $\quad ts * \pi \in \perp\!\!\!\perp \Rightarrow t * s.\pi \in \perp\!\!\!\perp$

but for every $t \in |A| \to |B|$ its η-expansion $\lambda x.tx \in |A \to B|$.

Of course, we have $|A \to B| = |A| \to |B|$ whenever $\perp\!\!\!\perp$ is also *closed under head reduction*, i.e. $\perp\!\!\!\perp \ni p \succeq q$ implies $q \in \perp\!\!\!\perp$.

Consistency is an Issue

If the pole $\perp\!\!\!\perp$ is empty the only truth values are Λ and $\emptyset$, i.e. we obtain the usual 2-valued classical logic.
However, if the pole is nonempty, i.e. $t * \pi \in \perp\!\!\!\perp$, then $|\perp| = \Pi^{\perp}$ is inhabited by $\mathsf{k}_\pi t$ since for all $\pi' \in \Pi$ we have

$$\mathsf{k}_\pi t * \pi' \succeq \mathsf{k}_\pi * t.\pi' \succeq t * \pi \in \perp\!\!\!\perp$$

For this reason we have defined A to be valid iff $|A|$ is inhabited by a quasi-proof.
Still one has to ensure that

$$\forall t \in \mathsf{QP}. \exists \pi \in \Pi.\ t * \pi \notin \perp\!\!\!\perp$$

which is achieved by

The Thread Model

Choose a bijection $t \mapsto \pi_t$ between QP and Π_0. The **thread** of $t \in$ QP is the set

$$\vartheta_t = \{t' * \pi' \mid t * \pi_t \succeq t' * \pi'\}$$

and we define $\perp\!\!\!\perp$ as the complement of the union of all threads.

Thus for all $t \in$ QP we have $t * \pi_t \notin \perp\!\!\!\perp$ from which it follows that $|\bot| \cap \mathrm{QP} = \emptyset$ as required by consistency.

The Thread Model - An Example

Put $\omega \equiv (\lambda x.xx)(\lambda x.xx)$. Let 0 and 1 stand for $\lambda x.\lambda y.x$ and $\lambda x.\lambda y.y$, respectively. Write ω_i for ωi.

If $t \in \Lambda \setminus \mathsf{QP}$ then it occurs in head position in at most one thread.

Lemma For every stack π we have $\omega_0 \mathsf{k}_\pi \in |\bot|$ or $\omega_1 \mathsf{k}_\pi \in |\bot|$

Proof. If both terms do not realize $\bot$ then they occur in head position in the same thread. This is impossible since ω loops.

The Classical Realizability Tripos

We obtain a boolean tripos P (over $\mathbf{Set}$) when putting

$$\Sigma = \mathcal{P}(\Pi)$$

and

$$\varphi \vdash_I \psi \quad \text{iff} \quad \exists t \in \mathsf{QP}.\forall i \in I.\forall s \in |\varphi_i|.\ ts \in |\psi_i|$$

for $\varphi, \psi \in \Sigma^I$.

NB It is easy to see that $\varphi \vdash_I \psi$ iff $\exists t \in \mathsf{QP}.\forall i \in I.\ t \in |\varphi_i \to \psi_i|$.

Verification

Proof (Sketch) :
Propositional connectives are interpreted as expected and logical laws are realized in a uniform way.
Quantification along projections $\pi : J \times I \to J$ is given by $\forall_\pi(\varphi)_j = \left(\bigcap_{i\in I} |\varphi_{ji}|\right)^{\perp}$.
The generic truth predicate is given by id_Σ. Thus we have quantification over predicates available.
We define equality à la Leibniz as $\mathtt{eq}_I(i,j) = \forall p{:}\Sigma^I.p(i) \to p(j)$.
Then for $u : J \to I$ universal quantication along u is given by $\forall_u(\varphi)_i = \forall j{:}J\ \mathtt{eq}_I(u(j),i) \to \varphi_j$. That it is right adjoint to u^* is proved by reasoning in the internal logic of P using that $\mathtt{eq}_I(i,j) \to \varphi_i \to \varphi_j$ is uniformly realizable. BCC holds on the nose.

A Simplification of Equality

Using classical logic one can show that negation of equality can be reformulated equivalently as

$$\neg\mathsf{eq}_I(i,j) = \{\pi \in \Pi \mid i = j\}$$

i.e. $\neg\mathsf{eq}_I(i,i) = \bot$ and $\neg\mathsf{eq}_I(i,j) = \top$ if $i \neq j$.

Assemblies

For a tripos P over $\mathbf{Set}$ one may consider the full subcategory of $\mathbf{Set}[\mathrm{P}]$ on **assemblies**, i.e. subobjects of some $\Delta(I)$, denoted as $\mathbf{Asm}(\mathrm{P})$. This category is always regular and $\mathbf{Set}[\mathrm{P}]$ is obtained from it by "adding quotient of equivalence relations".

If the tripos comes from a pca $\mathbb{A}$ the ensuing category $\mathbf{Asm}(\mathbb{A})$ of assemblies looks fairly concrete and hosts models of type theory.

Its objects are pairs $X = (|X|, E_X)$ where $|X|$ is a set and $E_X : |X| \to \mathcal{P}_{\geq 1}(\mathbb{A})$. A morphism from X to Y is simply a function $f : |X| \to |Y|$ which is tracked by some $a \in \mathbb{A}$, i.e.

$$\forall x \in |X|. \forall b \in E_X(x).\, ab \in E_Y(f(x))$$

What about assemblies for classical realizability?

Lack of Existence Property

In realizability over a pca if φ is a predicate on an assembly X and $\exists x{:}X.\varphi(x)$ there is a *global element* $a : 1 \to X$ such that $\varphi(a)$ holds. This is the reason why morphisms between assemblies are given as maps between their underlying sets.

Krivine has shown that this fails in classical realizability already for $\Delta(2)$ for the predicate

$$\varphi(x) \equiv \neg\mathsf{eq}(x,0) \wedge \neg\mathsf{eq}(x,1)$$

since $\neg\forall x{:}\Delta(2)\,(\neg\mathsf{eq}(x,0) \to \neg\mathsf{eq}(x,1) \to \bot)$ is realized by $\lambda f.\mathsf{cc}\lambda k.f(\omega_0 k)(\omega_1 k)$ though non of $\varphi(0)$ and $\varphi(1)$ are realizable.

Is Unique Existence always witnessed?
This would suffice for getting managable assemblies!

Generalisations

Since one wants to iterate the construction of classical realizability models one needs a general notion of **realizability structures**.

Krivine has developed one based on a large set of combinators which allows him to define λ-abstraction in such a way such that $(\lambda x.t)s \succeq t[s/x]$. For this everything goes through as above.

I have developed a variant based on the combinators K and S where $(\lambda x.t)s \succeq t[s/x]$ does not hold. Still it gives rise to a tripos but verification is more cumbersome. One has to go through an order pca with a filter which gives an equivalent indexed poset.