Probability and Nondeterminsim in Domain Theory

Part I

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Logic and Interaction 2012

Issues

- Representing probabilities in domain theory

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- Combining probability with nondeterminsm

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- Representing probabilities in domain theory
- Combining probability with nondeterminsm
- Troubles and solutions

DCPO

A directed complete partial order (DCPO)

- a partial order: $(D, \sqsubseteq)$
- with least upper bound of directed sets: $\sup K$

A continuous domain is a DCPO with a notion of approximation and a "basis"

Every element is the sup of elements of the basis

Continuous function: monotone and sup-preserving

Scott topology

Scott topology:

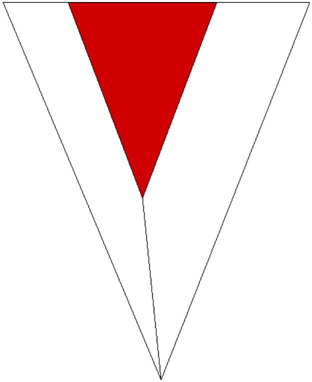
- The *closed sets* are the downward closed, sup-closed sets
- The *open sets* are the upward closed, unattainable sets

Open sets: "observable properties", "guarantee properties"
Continuous function: the same!

Separation axiom: only T_0

Example

Finite and infinite sequences of states Σ^{∞}
Infinite sequences approximated by finite sequences
Basic open: cones



definition

A *continuous valuation* on a topological space (X,σ)
A function $\nu : \sigma \to [0,1]$ satisfying

- (Strict and Normalised) $\nu(\emptyset) = 0, \nu(X) = 1$
- (Monotone) $U \sqsubseteq V \implies \nu(U) \leq \nu(V)$
- (Modular) $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$
- (Continuous)$\nu\left(\bigcup \mathscr{X}\right) = \sup_{U \in \mathscr{X}} \nu(U)$

Probabilistic choice

Probabilistic choice

$$(\nu \oplus_p \xi)(U) = p\nu(U) + (1-p)\xi(U)$$

Generalised probabilistic choice, if $\sum_i p_i = 1$

$$\bigoplus_i p_i \nu_i$$

Dirac's delta:

$$\delta_x(U) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}$$

A *simple* valuation:

$$\bigoplus_{x \in Y} p_x \delta_x$$

Jones and Plotkin

A continuous valuation on a DCPO D is a continuous valuation on its Scott topology. The set $\mathcal{V}(D)$ of continuous valuations on D ordered pointwise is again a DCPO

Theorem (Jones and Plotkin)

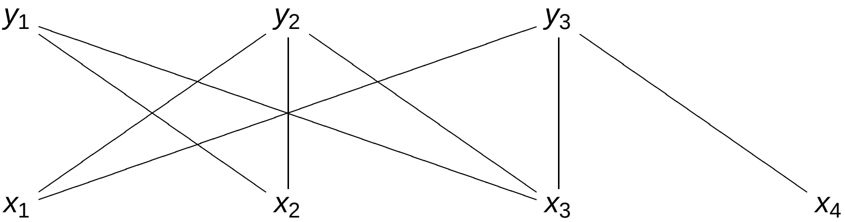
If D is a continuous domain, $\mathcal{V}(D)$ is a continuous domain with basis the set of simple valuations

A functor $\mathcal{V}$: **CONT** $\to$ **CONT**

Theorem (Splitting Lemma)

Let $\nu := \sum_{x \in X} r_x \delta_x$ *and* $\xi := \sum_{y \in Y} s_y \delta_y$ *be two simple valuations. We have that* $\nu \sqsubseteq \xi$ *if and only if there exist "transport numbers"* $t_{x,y}$ *such that*

- $\sum_{y \in Y} t_{x,y} = r_x$
- $\sum_{x \in X} t_{x,y} = s_y$
- $t_{x,y} > 0 \Longrightarrow x \sqsubseteq y$



Equations

$\oplus_p : \mathscr{V}(D) \times \mathscr{V}(D) \to \mathscr{V}(D)$ is continuous

- $A \oplus_p B = B \oplus_{1-p} A$
- $(A \oplus_p B) \oplus_q C = A \oplus_{pq} (B \oplus_{q(1-p)/1-pq} C)$
- $A = A \oplus_p A$
- $A = A \oplus_1 B$

Freeness

Theorem

The continuous valuations is the free domain-algebra for the theory

Therefore the functor $\mathscr{V}$ is in fact a monad in **CONT**

Jung and Tix

Main trouble: **CONT** is not cartesian closed
No cartesian closed categories of continuous domains is known that is closed under $\mathscr{V}$
Consequence: no semantics of functional languages

Hoare

The Hoare powerdomain of D

$$\mathscr{H}(D) = \{\emptyset \neq O \subseteq D \mid O \text{ Scott closed}\}$$

It models "angelic" nondeterminism

- $A \cup B = B \cup A$
- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cup A = A$
- $A \subseteq A \cup B$

$\mathscr{H}$ is the free algebra, thus a monad

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- $(A \cup B) \oplus_p C = (A \oplus_p C) \cup (B \oplus_p C)$

Convexity

They look like sets of valuations!! $\mathscr{H} \circ \mathscr{V}$

$$
\begin{aligned}
\{\nu\} \cup \{\mu\} &= \{\nu \cup \mu\} \oplus_p \{\nu \cup \mu\} \\
&= \{\nu \oplus_p \nu\} \cup \{\nu \oplus_p \mu\} \cup \{\mu \oplus_p \nu\} \cup \{\mu \oplus_p \mu\} \\
&= \{\nu\} \cup \{\nu \oplus_p \mu\} \cup \{\mu \oplus_p \nu\} \cup \{\mu\}
\end{aligned}
$$

They must be *convex* sets!

Dropping an equation

- $A \oplus_p B = B \oplus_{1-p} A$
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Coin flipping adds information

Dropping an equation

Other possibilities too

- $A \sqsupseteq A \oplus_p A$
- Drop the equation altogether

Finite indexed valuations

An abstrac basis

- X a basis of D
- I a finite set (indices)
- a function $f : I \to X$
- a function $prob : I \to [0, 1]$ such that $prob(i) > 0$ and $\sum_I prob(i) = 1$

Notation:
$(x_i, p_i)_{i \in I}$

Quotienting

The name and the order of the indices does not matter

$$(x_i, p_i)_{i \in I} \sim (y_j, q_j)_{j \in J}$$

if and only if there exists a bijection $h : I \to J$ such that

$$\forall i \in I.\ y_{h(i)} = x_i$$

$$\forall i \in I.\ q_{h(i)} = p_i$$

Probabilistic choice

$$(f_1, prob_1) \oplus_p (f_2, prob_2) =$$

$$(f_1 \uplus f_2, p \cdot prob_1 \uplus (1-p) \cdot prob_2)$$

Probabilistic choice

$$(x_i, q_i)_{i \in I} \oplus_p (x_j, q_j)_{j \in J} =$$

$$(x_h, p_h q_h)_{h \in I \uplus J}$$

where

$$p_h = \begin{cases} p & \text{if } h \in I \\ (1-p) & \text{if } h \in J \end{cases}$$

Ordering indexed valuations

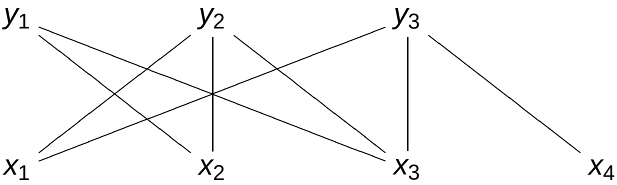
The order: an indexed splitting lemma

$$(x_i, p_i)_{i \in I} \sqsubseteq (y_j, q_j)_{j \in J}$$

if there a surjective function $h : J \to I$ such that

- $\sum_{h(j)=i} q_j = p_i$.
- $x_{h(j)} \sqsubseteq y_j$

The q_j are the “transport numbers” $t_{h(j),j}$



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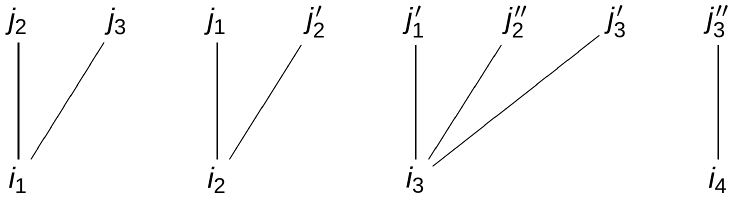

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$$A \sqsubseteq A \oplus_p A$$

Free Completion

Freely add all the sups

- A continuous domain $\mathscr{I}\mathscr{V}(D)$
- A functor
- The free algebra
- A monad

Everything works!

$\mathscr{H} \circ \mathscr{I}\mathscr{V}$ is a monad

Flattening

Forgetting indices:

$$\mathit{flat} : \mathscr{I}\mathscr{V}(D) \to \mathscr{V}(D)$$

- continuous
- surjective
- a natural transformation $\mathscr{I}\mathscr{V} \to \cdot\mathscr{V}$

Still problems

BUT:

- it is not known whether $\mathscr{I}\mathscr{V}$ preserves any cartesian closed categories of continuous domains
- There is no concrete representation for these sups!