Rewriting and homotopy,
Notes prepared for
Week 5 of Logic and Interactions 2012
Algebra and computation
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Introduction

Warning:

These notes for a mini-course in week 5 of LI 2012, and have been constructed from the main body of the much larger Menagerie notes, [110]. The method used to prepare them has been to delete sections that were not more or less necessary for this course and then to add in new material. (There will be some loose ends therefore, and missing links. These will be given with a ? as the Latex refers to the original cross reference.)

If you want to follow up some of the ideas that lead out of these notes, just look at the version of the Menagerie available on the nLab, [105], and if that does not have the relevant chapter, just ask me! (Beware, the full present version is already over 870 pages in length, so, please, don't print too many copies!!!)

There are several points to make. As in the full Menagerie notes, there are no exercises as such, but at various points if a proof could be expanded, or is **left to the reader**, then, yes, bold face will be used to suggest that that is a useful place for more input from the reader. In lots of places, reading the details is not that efficient a way of getting to grips with the calculations and ideas, and there is no substitute for **doing it yourself**. That being said guidance as to how to approach the subject will often be given.

Almost needless to say, there are things that have not been discussed here (or in the Menagerie itself), and suggestions for additional material are welcome. Better still would be for the suggestions to materialise into new entries on the nLab.

Introduction to the notes

The aim of these notes is to provide some background material for discussions of homotopy theory, simplicial group methods, algebraic models for n-types, crossed modules, some combinatorial group theory and to visit some other parts of low dimensional topology, useful for rewriting, initially in the context of combinatorial group theory. We then will go over to a subject dear to me: homotopy coherence, looking at it as both an exercise in applying rewriting, but also for its applications. Finally we will try to move away from the group case towards more general rewrite systems and thus towards 'Combinatorial category theory'.

The notes will give more than I will cover in the talks and before the start the plan for the talks is approximately as follows:

Lectures I. and II. Some combinatorial group theory and low dimensional homotopy: Presentations, identities among relations, crossed modules, and crossed resolutions. Homological and homotopical syzygies. Higher generation by subgroups (Abels and Holz). Examples.

Lecture III. An introduction to homotopy coherence and its rewriting aspect. Homotopy coherence and the resolution of a category. Examples of homotopy coherent diagrams and the homotopy coherent nerve. Quasi-categories. The link with rewriting.

Lecture IV. How to adapt away from the group theory case... brief discussion of directed homotopy, polygraphs etc. and how to work with rewriting and syzygies in the non-group case, some pointers to combinatorial category theory.

Tim Porter, Anglesey, 2012.

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Chapter 1

Preliminaries

1.1 Groups and Groupoids

Before launching into crossed modules, we need a word on groupoids. By a groupoid, we mean a small category in which all morphisms are isomorphisms. (If you have not formally met categories then do not worry, the idea will come through without that specific formal knowledge, although a quick glance at Wikipedia for the definition of a category might be a good idea at some time soon. You do not need category theory as such at this stage.) These groupoids typically arise in three situations (i) symmetry objects of a fibered structure, (ii) equivalence relations, and (iii) group actions. It is worth noting that several of the initial applications of groups were thought of, by their discoverers, as being more naturally this type of groupoid structure.

For the first, assume we have a family of sets $\{X_a : a \in A\}$. Typically we have a function $f : X \to A$ and $X_a = f^{-1}a$ for $a \in A$. We form the symmetry groupoid of the family by taking the index set, A, as the set of objects of the groupoid, $\mathcal{G}$, and, if $a, a' \in A$, then $\mathcal{G}(a, a')$, the set of arrows in our symmetry groupoid from a to a', is the set $Bijections(X_a, X_{a'})$. This $\mathcal{G}$ will contain all the individual symmetry groups / permutation groups of the various X_a, but will also record comparison information between different X_as.

Of course, any group is a groupoid with one object and if $\mathcal{G}$ is any groupoid, we have, for each object a of $\mathcal{G}$, a group $\mathcal{G}(a, a)$, of arrows that start and end at a. This is the 'automorphism group', $\mathsf{aut}_{\mathcal{G}}(a)$, of a within $\mathcal{G}$. It is also referred to as the vertex group of $\mathcal{G}$ at a, and denoted $\mathcal{G}(a)$. This later viewpoint and notation emphasise more the combinatorial, graph-like side of $\mathcal{G}$'s structure. Sometimes the notation $G[1]$ may be used for $\mathcal{G}$ as the process of regarding a group as a groupoid is a sort of 'suspension' or 'shift'. It is one aspect of 'categorification', cf. Baez and Dolan, [7].

That combinatorial side is strongly represented in the second situation, equivalence relations. Suppose that R is an equivalence relation on a set X. Going back to basics, R is a subset of $X \times X$ satisfying:

(a) if $a, b, c \in X$ and (a, b) and $(b, c) \in R$, then $(a, c) \in R$, i.e., R is transitive;

(b) for all $a \in X$, $(a, a) \in R$, alternatively the diagonal $\Delta \subseteq R$, i.e., R is reflexive;

(c) if $a, b \in X$ and $(a, b) \in R$, then $(b, a) \in R$, i.e., R is symmetric.

Two comments might be made here. The first is 'everyone knows that!', the second 'that is not the usual order to put them in! Why?'

It is a well known, but often forgotten, fact that from R, you get a groupoid (which we will denote by $\mathcal{R}$). The objects of $\mathcal{R}$ are the elements of X and $\mathcal{R}(a,b)$ is a singleton if $(a,b) \in \mathcal{R}$ and is empty otherwise. (There is really no need to label the single element of $\mathcal{R}(a,b)$, when this is non empty, but it is sometimes convenient to call it (a,b) at the risk of over using the ordered pair notation.) Now transitivity of R gives us a composition function: for $a, b, c \in X$,

$$\circ : \mathcal{R}(a,b) \times \mathcal{R}(b,c) \to \mathcal{R}(a,c).$$

(Remember that a product of a set with the empty set is itself always empty, and that for any set, there is a unique function with domain $\emptyset$ and codomain the set, so checking that this composition works nicely is slightly more subtle than you might at first think. This *is* important when handling the analogues of equivalence relations in other categories., then you cannot just write $(a,b)\circ(b,c) = (a,c)$, or similar, as 'elements' may not be obvious things to handle.) Of course this composition *is* associative, but if you have not seen the verification, it is important to think about it, looking for subtle points, especially concerning the empty set and empty function and how to do the proof without 'elements'.

This composition makes $\mathcal{R}$ into a category, since (a) gives the existence of identities for each object. ($Id_a = (a,a)$ in 'elementary' notation.) Finally (c) shows that each (a,b) is invertible, so $\mathcal{R}$ is a groupoid. (You now see why that order was the natural one for the axioms. You cannot prove that (a,a) is an identity until you have a composition, and similarly until you have identities, inverses do not make sense.) We may call $\mathcal{R}$, the groupoid of the equivalence relation R.

This shows how to think of R as a groupoid, $\mathcal{R}$. The automorphism groups, $\mathcal{R}(a)$, are all singletons as sets, so are trivial groups. Conversely any groupoid, $\mathcal{G}$, gives a diagram

$$\begin{tikzcd} Arr(\mathcal{G}) \arrow[r, shift left=2, "s"] \arrow[r, shift left=0.5, "t"'] & Ob(\mathcal{G}) \arrow[l, shift left=1.5, "i"] \end{tikzcd}$$

with $s =$ 'source', $t =$ 'target'. It thus gives a function

$$Arr(\mathcal{G}) \xrightarrow{(s,t)} Ob(\mathcal{G}) \times Ob(\mathcal{G}) \ .$$

The image of this function is an equivalence relation as is easily checked. We will call this equivalence relation R for the moment. If $\mathcal{G}$ is a groupoid such that each $\mathcal{G}(a)$ is a trivial group, then each $\mathcal{G}(a,b)$ has at most one element (check it), so (s,t) is a one-one function and it is then trivial to note that $\mathcal{G}$ is isomorphic to the groupoid of the equivalence relation, R.

We have looked at this simple case in some detail as in applications of the basic ideas, especially in algebraic geometry, arguments using elements are quite tricky to give and the initial intuition coming from this set-based case can easily be forgotten.

The third situation, that of group actions, is also a common one in algebra and algebraic geometry. Equivalence relations often come from group actions. If G is a group and X is a G-set with (left) G-action,

$$\begin{array}{ccc} G \times X & \longrightarrow & X \quad , \\ (g,x) & & g \cdot x \end{array}$$

(i.e., a function $act(g,x) = g \cdot x$, which must satisfy the rules $1 \cdot x = x$ and for all $g_1, g_2 \in G$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, a sort of associativity law), then we get a groupoid $\mathcal{A}ct_G(X)$, that will be called the *action groupoid* of the G-set, as follows:

- the objects of $\mathcal{A}ct_G(X)$ are the elements of X;
- if $a, b, \in X$,
$$\mathcal{A}ct_G(X)(a,b) \cong \{g \mid g \cdot a = b\}.$$

An important word of caution is in order here. Logical complications can occur here if $\mathcal{A}ct_G(X)(a,b)$ is set equal to $\{g \mid g \cdot a = b\}$, since then a g can occur in several different 'hom-sets'. A good way to avoid this is to take
$$\mathcal{A}ct_G(X)(a,b) = \{(g,a) \mid g \cdot a = b\}.$$
This is a non-trivial change. It basically uses a disjoint union, but although very simple, it is fundamental in its implications. We could also do it by taking $Arr_{\mathcal{G}}(X) = G \times X$ with source and target maps $s(g,x) = x$, $t(g,x) = g \cdot x$. (It is **useful**, if you have not seen this before, to see how the various parts of the definition of an action match with parts of the structural rules of a groupoid. This is important as it indicates how, much later on, we will relax those rules in various ways.)

We will sometimes use the notation, $G \curvearrowright X$, when discussing a left action of a group G on X.

In a groupoid, G, we say two objects, x and y are in the same *connected component* of G, if $G(x,y)$ is not empty. This gives an equivalence relation on the set of objects of G, as you **can easily check**. The equivalence classes re called the *connected components* of G and the set of connected components is usually denoted $\pi_0(G)$, by analogy with the usual notion for the set of connected components of a topological space.

We have not discussed morphisms of groupoids. These are straightforward to define and to work with. Together groupoids and the morphisms between them form a *category*, the *category of groupoids*, which will be denoted *Grpds*.

(As we introduced structures of various types, we will usually introduce a corresponding form of morphism and it will be rare that the resulting 'context' of objects and morphisms does not form a category. It is important to look up the definition of categories and functors, but *for the moment* you will not need to know any 'category theory' to read the notes. It will suffice to get to grips with that as we go further and have good motivating examples for what is needed.)

Most of the concepts that we will be handling in what follows exist in many-object, groupoid versions as well as single-object, group based ones. For simplicity we will often, but not always, give concepts in the group based form, and will leave the other many-object form 'to the reader'. The conversion is usually not that difficult.

For more details on the theory of groupoids, the best two sources are Ronnie Brown's book, [27] or Phil Higgins' monograph, now reprinted as [71].

1.2 A very brief introduction to cohomology

Partially as a case study, at least initially, we will be looking at various constructions that relate to group cohomology. Later we will explore a more general type of (non-Abelian) cohomology, including ideas about the non-Abelian cohomology of spaces, but that is for later. To start with we will look at a simple group theoretic problem that will be used for motivation at several places in what follows. Much of what is in books on group cohomology is the Abelian theory, whilst we will be looking more at the non-Abelian one. If you have not met cohomology at all, take a look at the Wikipedia entries for group cohomology. You may not understanding everything, but there are ideas there that will recur in what follows, and some terms that are described there or on linked entries, that will be needed later.

1.2.1 Extensions.

Given a group, G, an extension of G by a group K is a group E with an epimorphism $p : E \to G$ whose kernel is isomorphic to K (i.e. a short exact sequence of groups

$$\mathcal{E} : 1 \to K \to E \xrightarrow{p} G \to 1.$$

As we asked that K is isomorphic to $Ker\, p$, we could have different groups E perhaps fitting into this, yet they would still be essentially the same extension. We say two extensions, $\mathcal{E}$ and $\mathcal{E}'$, are equivalent if there is an isomorphism between E and E' compatible with the other data. We can draw a diagram

$$\begin{tikzcd} \mathcal{E} \arrow[d] & 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\cong"'] & G \arrow[r] \arrow[d, "="] & 1 \\ \mathcal{E}' & 1 \arrow[r] & K \arrow[r] & E' \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

A typical situation might be that you have an unknown group E' that you suspect is really E (i.e. is isomorphic to E). You find a known normal subgroup K of E is isomorphic to one in E' and that the two quotient groups are isomorphic,

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "\cong"'] & E \arrow[r] \arrow[d, dashed, "?"'] & G \arrow[r] \arrow[d, "\cong"] & 1 \\ 1 \arrow[r] & K' \arrow[r] & E' \arrow[r] & G' \arrow[r] & 1 \end{tikzcd}$$

(But always remember, isomorphisms compare snap shots of the two structures and once chosen can make things more ‘rigid’ than perhaps they really ‘naturally’ are. For instance, we might have G a cyclic group of order 5 generated by an element a, and G' one generated by b. ‘Naturally’ we choose an isomorphism $\varphi : G \to G'$ to send a to b, but why? We could have sent a to any non-identity element of G' and need to be sure that this makes no difference. This is not just ‘attention to detail’. It can be very important. It stresses the importance of $Aut(G)$, the group of automorphisms of G in this sort of situation.)

A simple case to illustrate that the extension problem is a valid one, is to consider $K = C_3 = \langle a \mid a^3\rangle$, $G = C_2 = \langle b \mid b^2\rangle$.

We could take $E = S_3$, the symmetric group on three symbols, or alternatively D_3 (also called D_6 to really confuse things, but being the symmetry group of the triangle). This has a presentation $\langle a, b \mid a^3, b^2, (ab)^2\rangle$. But what about $C_6 = \langle c \mid c^6\rangle$? This has a subgroup $\{1, c^2, c^4\}$ isomorphic to K and the quotient is isomorphic to G. Of course, S_3 is non-Abelian, whilst C_6 is. The presentation of C_6 needs adjusting to see just how similar the two situations are. This group also has a presentation $\langle a, b \mid a^3, b^2, aba^{-1}b\rangle$, since we can deduce $aba^{-1}b = 1$ from $[a, b] = 1$ and $b^2 = 1$ where in terms of the old generator c, $a = c^2$ and $b = c^3$. So there is a presentation of C_3 which just differs by a small ‘twist’ from that of S_3.

How could one be sure if S_3 and C_6 are the 'only' groups (up to isomorphism) that we could put in that central position? Can we classify all the extensions of G by K?

These extension problems were one of the impetuses for the development of a 'cohomological' approach to algebra, but they were not the only ones.

1.2.2 Invariants

Another group theoretic input is via group representation theory and the theory of invariants. If G is a group of $n \times n$ invertible matrices then one can use the simple but powerful tools of linear algebra to get good information on the elements of G and often one can tie this information in to some geometric context, say, by identifying elements of G as leaving invariant some polytope or pattern, so G acts as a subgroup of the group of the symmetries of that pattern or object.

If, therefore, we use the group $Gl(n, \mathbb{K})$ of such invertible matrices over some field $\mathbb{K}$, then we could map an arbitrary G into it and attempt to glean information on elements of G from the corresponding matrices. We thus consider a group homomorphism

$$\rho : G \to Gl(n, \mathbb{K}),$$

then look for nice properties of the $\rho(g)$. of course, ρ need not be a monomorphism and then we will loose information in the process, but in any case such a morphism will make G act (linearly) on the vector space $\mathbb{K}^n$. We could, more generally, replace $\mathbb{K}$ by a general commutative ring R, in particular we could use the ring of integers, $\mathbb{Z}$, and then replace $\mathbb{K}^n$ by a general *module*, M, over R. If $R = \mathbb{Z}$, then this is just an Abelian group. (If you have not formally met modules look up a definition. The theory feels very like that of vector spaces to start with at least, but as elements in R need not have inverses, care needs to be taken - you cannot cancel or divide in general, so $rx = ry$ does not imply $x = y$! Having looked up a definition, for most of the time you can think of modules as being vector spaces or Abelian groups and you will not be far wrong. We will shortly but briefly mention modules over a group algebra, $R[G]$, and that ring is not commutative, but again the complications that this does cause will not worry us at all.)

We can thus 'represent' G by mapping it into the automorphism group of M. This gives M the structure of a G-module. We look for invariants of the action of G on M - what are they? Suppose that G is some group of symmetries of some geometric figure or pattern, that we will call X, in $\mathbb{R}^n$, then for each $g \in G$, $gX = X$, since g acts by pushing the pattern around back onto itself. An invariant of G, considered as acting on M, or, to put it more neatly, of the G-module, M, is an element m in M such that $g.m = m$ for all $g \in G$. These form a submodule,

$$M^G = \{m \mid gm = m \text{ for all } g \in G\}.$$

Clearly, it will help in our understanding of the structure of G if we can calculate and analyse these modules of invariants. Now suppose we are looking at a submodule N of M, then N^G is a submodule of M^G and we can hope to start finding invariants, perhaps by looking at such submodules and the corresponding quotient modules, M/N. We have a short exact sequence

$$0 \to N \to M \to M/N \to 0,$$

but, although applying the (functorial) operation $(-)^G$ does yield

$$0 \to N^G \to M^G \to (M/N)^G,$$

the last map need not be onto so we may not get a short exact sequence and hence a nice simple way of finding invariants!

Example: Try $G = C_2 = \{1, a\}$, $M = \mathbb{Z}$, the Abelian group of integers, with G action, $a.n = -n$, and $N = 2\mathbb{Z}$, the subgroup of even integers, with the same G action. Now calculate the invariant modules M^G and N^G; they are both trivial, but $M/N \cong Z_2$, and ..., what is $(M/N)^G$ for this example?

The way of studying this in general is to try to to continue the exact sequence further to the right in some universal and natural way (via the theory of derived functors). This is what cohomology does. We can get a long exact sequence,

$$0 \to N^G \to M^G \to (M/N)^G \to H^1(G,N) \to H^1(G,M) \to H^1(G,M/N) \to H^2(G,N) \to \ldots.$$

But what are these $H^k(G,M)$ and how does one get at them for calculation and interpretation? In fact what is cohomology in general?

Its origins lie within Algebraic Topology as well as in Group Theory and that area provides some useful intuitions to get us started, before asking how to form group cohomology.

1.2.3 Homology and Cohomology of spaces.

Naively homology and cohomology give methods for measuring the holes in a space, holes of different dimensions yield generators in different (co)homology groups. The idea is easily seen for graphs and low dimensional simplicial complexes.

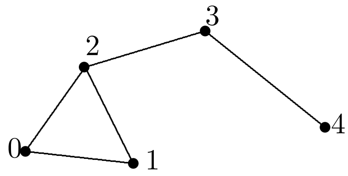
First we recall the definition of simplicial complex as we will need to be fairly precise about such objects and their role in relation to triangulations and related concepts.

Definition: A *simplicial complex*, K, is a set of objects, $V(K)$, called *vertices* and a set, $S(K)$, of finite non-empty subsets of $V(K)$, called *simplices*. The simplices satisfy the condition that if $\sigma \subset V(K)$ is a simplex and $\tau \subset \sigma$, $\tau \neq \emptyset$, then τ is also a simplex.

We say τ is a *face* of σ. If $\sigma \in S(K)$ has $p+1$ elements it is said to be a *p-simplex*. The set of p-simplices of K is denoted by K_p. The dimension of K is the largest p such that K_p is non-empty.

We will sometimes use the notation, $\mathcal{P}(X)$, for the power set of a set X, i.e., the set of subsets of X. Suppose that $X = \{0, \ldots, p\}$, then there is a simple example of a simplicial complex, known as the *standard abstract p-simplex*, $\Delta[n]$, with vertex set, $V(\Delta[n]) = X$ and with $S(\Delta[n]) = \mathcal{P}(X)\backslash\{\emptyset\}$, in other words all non-empty subsets of X are to be simplices. (If you have not met simplicial complexes before this is a **good example to work with working out** what it looks like and 'feels like' for $n = 0, 1, 2$ and 3. It is too regular to be general, so we will, below, see another example which is perhaps a bit more typical.

When thinking about simplicial complexes, it is important to have a picture in our minds of a triangulated space (probably a surface or similar, a wireframe as in computer graphics). The simplices are the triangles, tetrahedra, etc., and are determined by their sets of vertices. Not every set of vertices need be a simplex, but if a set of vertices does correspond to a simplex then all its non-empty subsets do as well, as they give the faces of that simplex. Here is an example:



Here $V(K) = \{0, 1, 2, 3, 4\}$ and $S(K)$ consists of $\{0, 1, 2\}$, $\{2, 3\}$, $\{3, 4\}$ and all the non-empty subsets of these. Note the triangle $\{0, 1, 2\}$ is intended to be solid, (but I did not work out how to do it on the Latex system I was using!)

Simplicial complexes are a natural combinatorial generalisation of (undirected) graphs. They not only have vertices and edges joining them, but also possible higher dimensional simplices relating paths in that low dimensional graph. It is often convenient to put a (total) order on the set $V(K)$ of vertices of a simplicial complex as this allows each simplex to be specified as a list $\sigma = \langle v_0, v_1, \ldots, v_n\rangle$ with $v_0 < v_1 < \ldots < v_n$, instead of as merely a set $\{v_0, v_1, \ldots, v_n\}$ of vertices. This, in turn, allows us to talk, unambiguously, of the k^{th} face of such a simplex, being the list with v_k omitted, so the zeroth face is $\langle v_1, \ldots, v_n\rangle$, the first is $\langle v_0, v_2, \ldots, v_n\rangle$ and so on.

Although strictly speaking different types of object, we tend to use the terms 'vertex' and '0-simplex' interchangeably and also use 'edge' as a synonym for '1-simplex'. We will usually write K_0 for $V(K)$ and may write K_1 for the set of edges of a graph, thought of as a 1-dimensional simplicial complex.

An abstract simplicial complex is a combinatorial gadget that models certain aspects of a spatial configuration. Sometimes it is useful, perhaps even necessary, to produce a topological space from that data in a simplicial complex.

Definition: To each simplicial complex K, one can associate a topological space called the *polyhedron* of K often also called or *geometric realisation* of K and denoted $|K|$.

This can be constructed by taking a copy $K(\sigma)$ of a standard topological p-simplex for each p-simplex of K and then 'gluing' them together according to the face relations encoded in K.

Definition: The *standard (topological) p-simplex* is usually taken to be the convex hull of the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{p+1}$ in $\mathbb{R}^{p+1}$, to represent each abstract p-simplex, $\sigma \in S(K)$, and then 'gluing' faces together, so whenever τ is a face of σ we identify $K(\tau)$ with the corresponding face of $K(\sigma)$. This space is usually denoted Δ^p.

There is a canonical way of constructing $|K|$ as follows: $|K|$ is the set of all functions from $V(K)$ to the closed interval $[0, 1]$ such that

- if $\alpha \in |K|$, the set
$$\{v \in V(K) \mid \alpha(v) \neq 0\}$$
is a simplex of K;

- for each $v \in V(K)$, $\sum_{\alpha\in V(K)} \alpha(v) = 1.$

We can put a metric d on $|K|$ by

$$d(\alpha,\beta) = \Big(\sum_{v\in V(K)} (p_v(\alpha) - p_v(\beta))^2\Big)^{\frac{1}{2}}.$$

This however gives $|K|$ as a subspace of $\mathbb{R}^{\#(V(K))}$, and so is usually of much higher dimension then might seem geometrically significant in a given context. For instance, the above example would be represented as a subspace of $\mathbb{R}^5$, rather than $\mathbb{R}^2$, although that is the dimension of the picture we gave of it.

Given two simplicial complexes K, L, then a function on the vertex sets, $f : V(K) \to V(L)$ is a *simplicial map* if it preserves simplices. (But that needs a bit of care to check out its exact meaning! ... **for you to do**. Look it up, or better try to see what the problem might be, try to resolve it yourself and then look it up!)

1.2.4 Betti numbers and Homology

One of the first sorts of invariant considered in what was to become Algebraic Topology was the family of Betti numbers. Given a simple shape, the most obvious piece of information to note would be the number of 'pieces' it is made up of, or more precisely, the number of *components*. The idea is very well known, at least for graphs, and as simplicial complexes are closely related to graphs, we will briefly look at this case first.

For convenience we will assume the vertices $V = V(\Gamma)$ of a given finite graph, Γ, are ordered, so for each edge e of Γ, we can assign a source $s(e)$ and a target $t(e)$ amongst the vertices. Two vertices v and w are said to be in the same component of Γ if there is a sequence of edges $e_1, \ldots, e_k$ of Γ joining them[1]. There are, of course, several ways of thinking about this, for instance, define a relation $\sim$ on V by : for each e, $s(e) \sim t(e)$. Extend $\sim$ to an equivalence relation on V in the standard way, then $v \sim w$ if and only if they are in the same component. The zeroth Betti number, $\beta_0(\Gamma)$, is the number of components of Γ.

The first Betti number, $\beta_1(\Gamma)$, somewhat similarly, counts the number of cycles of Γ. We have ordered the vertices of Γ, so have effectively also directed its edges. If e is an edge, going from u to v, (so $u < v$ in the order on Γ_0), we write e also for the path going just along e and $-e$ for that going backwards along it, then extend our notation so $s(-e) = t(e) = v$, etc. Adding in these 'negative edges' corresponds to the formation of the symmetric closure of $\sim$. For the transitive closure we need to concatenate these simple one-edge paths: if e' is an edge or a 'negative edge' from v to w, we write $e + e'$ for the path going along e then e'. Playing algebraically with s and t and making them respect addition, we get a 'pseudo-calculation' for their difference $\partial = t - s$:

$$\partial(e + e') = t(e + e') - s(e + e') = t(e) + t(e') - s(e) - s(e') = t(e') - s(e) = u - w,$$

since $t(e) = v = s(e')$. In other words, defined in a suitable way, we would get that ∂, equal to 'target minus source', applies nicely to paths as well as edges, so that, for instance, two vertices

[1] In fact here, the ordering we have assumed on the vertices complicates the exposition a little, but it is useful later on so will stick with it here.

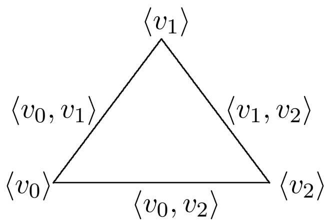
would be related in the transitive closure of $\sim$ if there was a 'formal sum' of edges that mapped down to their 'difference'. We say 'formal sum' as this is just what it is. We will need 'negative vertices' as well as 'negative edges'.

We set this up more formally as follows: Let
$C_0(\Gamma)$ = the set of formal sums, $\sum_{v\in\Gamma_0} a_v v$ with $a_v \in \mathbb{Z}$, the additive group of integers, (an alternative form is to take $a_v \in \mathbb{R}$.;
$C_1(\Gamma)$ = the set of formal sums, $\sum_{e\in\Gamma_1} b_e e$ with $b_e \in \mathbb{Z}$,
where Γ_1 denotes the set of edges of Γ, and $\partial : C_1(\Gamma) \to C_0(\Gamma)$ defined by extending additively the mapping given on the edges by $\partial = t - s$.

The task of determining components is thus reduced to calculating when integer vectors differ by the image of one in $C_1(\Gamma)$. The Betti number $\beta_0(\Gamma)$ is just the *rank* of the quotient $C_0(\Gamma)/Im(\partial)$, that is, the number of free generators of this commutative group. This would be exactly the dimension of this 'vector space' if we had allowed real coefficients in our formal sums not just integer ones.

Having reformulated components and $\sim$ in an algebraic way, we immediately get a pay-off in our determination of cycles. A cycle is a path which starts and ends at the same vertex; a path is being modelled by an element in $C_1(\Gamma)$, so a cycle is an element x in $C_1(\gamma)$ satisfying $\partial(x) = 0$. With this we have $\beta_1(\Gamma) = rank(Ker(\partial))$, a similar formulation to that for β_0. The similarity is even more striking if we replace the graph Γ by a simplicial complex K. We can then define in general and in any dimension p, $C_p(K)$ to be the commutative group of all formal sums $\sum_{\sigma\in K_p} a_\sigma \sigma$.

We next need to get an analogue of the $\partial = t - s$ formula. We want this to correspond to the boundary of the objects to which it is applied. For instance, if σ was the triangle / 2-simplex, $\langle v_0, v_1, v_2\rangle$, we would want $\partial\sigma$ to be $\langle v_1, v_2\rangle + \langle v_0, v_1\rangle - \langle v_0, v_2\rangle$, since going (clockwise) around the triangle, that cycle will be traced out:



If we write, in general, $d_i\sigma$ for the i^{th} face of a p-simplex $\sigma = \langle v_0, \ldots, v_p\rangle$, then in this 2-dimensional example $\partial\sigma = d_0\sigma - d_1\sigma + d_2\sigma$, changing the order for later convenience. This is the sum of the faces with weighting $(-1)^i$ given to $d_i\sigma$. This is consistent with $\partial = t - s$ in the lower dimension as $t = d_0$ and $s = d_1$. We can thus suggest that

$$\partial = \partial_p : C_p(K) \to C_{p-1}(K)$$

be defined on p-simplices by

$$\partial_p\sigma = \sum_{i=0}^{p} (-1)^i d_i\sigma,$$

and then extended additively to all of $C_p(K)$.

As an example of what this does, look at a square K, with vertices v_0, v_1, v_2, v_3, edges $\langle v_i, v_{i+1}\rangle$ for $i = 0, 1, 2$ and $\langle v_0, v_2\rangle$, and 2-simplices $\sigma_1 = \langle v_0, v_1, v_2\rangle$ and $\sigma_2 = \langle v_0, v_2, v_3\rangle$. As the square has these two 2-simplices, we can think of it as being represented by $\sigma_1 + \sigma_2$ in $C_2(K)$, then $\partial(\sigma_1 + \sigma_2) = \langle v_0, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_3\rangle - \langle v_0, v_3\rangle$, as the two occurrences of the diagonal $\langle v_0, v_2\rangle$ cancel out as they have opposite sign, and this is the path around the actual boundary of the square.

It is important to note that the boundary of a boundary is always trivial, that is, the composite mapping

$$C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} C_{p-2}(K)$$

is the mapping sending everything to $0 \in C_{p-1}(K)$.

The idea of the higher Betti numbers, $\beta_p(K)$, is that they measure the number of p-dimensional 'holes' in K. Imagine we has a tunnel-shaped hole through a space K, then we would have a cycle around the hole at one end of the tunnel and another around the hole at the other end. If we merely count cycles then we will get at least two such coming from this hole, but these cycles are linked as there is the cylindrical hole itself and that gives a 2 dimensional element with boundary the difference of the two cycles. In general, a *p-cycle* will be an element x of $C_p(K)$ with trivial boundary, i.e., such that $\partial x = 0$, and we say that two p-cycles x and x' are *homologous* if there is an element y in $C_{p+1}(K)$ such that $\partial y = x - x'$. The 'holes' correspond to classes of homologous cycles as in our tunnel.

The number of 'independent' cycle classes in the various dimensions give the corresponding Betti number. Using some algebra, this is easier to define rigorously, but, at the same time, the geometric insights from the vaguer description are important to try to retain. (They are not always put in a central enough position in textbooks!) This algebraic approach identifies $\beta_p(K)$ as the (torsion free) rank of a certain commutative group formed as follows: the p^{th} homology group of K is defined to be the quotient:

$$H_p(K) = \frac{Ker(\partial_p : C_p(K) \to C_{p-1}(K))}{Im(\partial_p : C_{p+1}(K) \to C_p(K))},$$

and then $\beta_p(K) = rank(H_p(K))$.

Thus far we have from K built a sequence of modules, $C(K)_n$, generated by the n-simplices of K and with homomorphisms $\partial_p : C_p(K) \to C_{p-1}(K)$ satisfying $\partial_{p-1}\partial_p = 0$.. (We abstract this structure calling it a *chain complex.* We will look at in more detail at several places later in these notes.)

Exercises: Try to investigate this homology in some very simple situations perhaps including some of the following:
(a) $V(K) = \{0, 1, 2, 3\}$, $S(K) = \mathcal{P}(V(K)) \setminus \{\emptyset, \{0, 1, 2, 3\}\}$. This is an empty tetrahedron so one expects one 3-dimensional hole., i.e., $\beta_3(K) = 1$ but the others are zero.
(b) $\Delta[2]$ is the (full) triangle and $\partial\Delta[2]$ its boundary, so is an empty triangle. Find the homology of $\partial\Delta[2] \times \partial\Delta[2]$, which is a triangulated torus.
(c) Find the homology of $\Delta[1] \times \partial\Delta[2]$, which is a cylinder.

Note, it is up to you to find the meaning of product in this context. Remember the discussion of the square, above, which is, of course $\Delta[1] \times \Delta[1]$.

Often cohomology is more use than homology. Starting with K and a module M work out $C^n(K, M) = Hom(C(K)_n, M)$. Now the boundary maps increase (upper) degree by one. The cohomology is $H^n(K, M) = Ker\,\partial^n/Im\,\partial^{n-1}$. Again this measures 'holes' detectable by M! What does that mean? The cohomology groups are better structured than the homology ones, but how are these invariants be interpreted?

A simplicial map, $f : K \to L$, will induce a map on cohomology groups. Try it! We can equally well do this for chain or 'cochain complexes'. There is a notion of chain map between chain complexes, say, $\varphi : C \to D$ and such a map will induce maps on both homology ad cohomology. Of special interest is when the induced maps are isomorphisms. The chain map is then called a *quasi-isomorphism*.

1.2.5 Interpretation

The question of interpretation is a very crucial question, but, rather than answering it now, we will return to the cohomology of groups. The terminology may seem a bit strange. Here we have been talking about measuring holes in a space, so how does that relate to groups. The idea is that one builds a space from a group in such a way as the properties of the space reflect those of the group in some sense. The simplest case of this is an Eilenberg-MacLane space, $K(G, 1)$. The defining property of such a space is that its fundamental group is G whilst all other homotopy groups are trivial. Eilenberg and Maclane showed that however such a space was constructed its cohomology could be got just from G itself and that cohomology was related with the extension problem and the invariant module problem. Their method was to build a chain complex that would copy the structure of the chain complex on the $K(G, 1)$. This chain complex, *the bar resolution*, was very important because although in the group case there was an alternative route via the topological space $K(G, 1)$, for many other types of algebraic system (Lie algebras, associative algebras, commutative algebras, etc.), the analogous basic construction could be used, and in those contexts no space was available. Thus from G, we want to construct a nice chain complex directly. The construction is reasonably simple. It gives a natural way of getting a chain complex, but it does not exploit any particular features of the group so if the group is infinite, the modules will be infinitely generated, which will occupy us later, as we use insights from combinatorial group theory to construct smaller models for equivalent resolutions, and better still look at 'crossed' versions.

For the moment we just need the definition (adapted from the account given in Wikipedia):

1.2.6 The bar resolution

The input data is a group G and a module M with a left G-action (i.e., a left G-module).

For $n \geq 0$, we let $C^n(G, M)$ be the group of all *functions* from the n-fold product G^n to M:

$$C^n(G, M) = \{\varphi : G^n \to M\}$$

This is an Abelian group; its elements are called the *n-cochains*. We further define group homomorphisms

$$\partial^n : C^n(G, M) \to C^{n+1}(G, M)$$

by

$$\begin{aligned}\partial^n(\varphi)(g_0, \ldots, g_n) \quad = \quad & g_0 \cdot \varphi(g_1, \ldots, g_n) \\ & \quad + \sum_{i=0}^{n-1} (-1)^{i+1} \varphi(g_0, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) \\ & \qquad + (-1)^{n+1} \varphi(g_0, \ldots, g_{n-1})\end{aligned}$$

These are known as the *coboundary homomorphisms*. The crucial thing to check here is $\partial^{n+1} \circ \partial^n = 0$, thus we have a chain complex and we can 'compute' its cohomology. For $n \geq 0$, define the group of n-cocycles as:

$$Z^n(G, M) = Ker\, \partial^n$$

and the group of n-coboundaries as

$$\begin{cases} B^0(G, M) = 0 \\ B^n(G, M) = Im(\partial^{n-1}) & n \geq 1 \end{cases}$$

and

$$H^n(G, M) = Z^n(G, M)/B^n(G, M).$$

Thinking about this topologically, it is as if we had constructed a sort of space / simplicial complex, K, out of G by taking $K_n = G^n$. We will see this idea many times later on. This cochain complex is often called the *bar resolution*. It exists in a normalised and a unnormalised form. This is the unnormalised one. It can also be constructed via a chain complex, sometimes denoted βG, so that this $C(G, M)$ is formed by taking $Hom(\beta G, M)$, in a suitable sense.

There are lots of properties that are easy to check here. Some will be suggested as exercises for you to do. For others, you can refer to some of the standard textbooks that deal with introductions to group cohomology, for instance, K. Brown's [25].

One further point is that this cohomology used a module, and so encodes 'commutative' or Abelian information. We will be also looking at the non-Abelian case.

Before we leave this introduction to cohomology, it should be mentioned that in the topological case, if we do not have a simplicial complex to start with, we either use the singular complex (see next section) which is a simplicial set and not a simplicial complex, but the theory extends easily enough, or we use open covers of the space to build a system of simplicial complexes approximating to the space. We will see this later as Čech cohomology. This is most powerful when the module M of coefficients is allowed to vary over the various points of the space. For this we will need the notion of sheaf, which will be discussed in some detail later.

1.3 Simplicial things in a category

1.3.1 Simplicial Sets

Simplicial objects are extremely useful. Simplicial sets extend ideas of simplicial complexes in a neat way. They combine a reasonably simple combinatorial definition with subtle algebraic properties. Their original construction was motivated in algebraic topology by the singular complex of a space.

If X is a topological space, $Sing(X)$ denotes the collection of sets and mappings defined by

$$Sing(X)_n = Top(\Delta^n, X), \qquad\qquad n \in \mathbb{N},$$

where Δ^n is the usual topological n-simplex given, for example, by

$$\{\underline{x} \in \mathbb{R}^{n+1} \mid \sum x_i = 1; \;\; \text{all } x_i \geq 0\}.$$

There are inclusion maps $\delta_i : \Delta^{n-1} \to \Delta^n$ and 'squashing' maps $\sigma_i : \Delta^{n+1} \to \Delta^n$ and these induce the face maps,

$$d_i : Sing(X)_n \to Sing(X)_{n-1}, \qquad 0 \leq i \leq n,$$

and degeneracy maps,

$$s_i : Sing(X)_n \to Sing(X)_{n+1}, \qquad 0 \leq i \leq n.$$

These satisfy the simplicial identities,

$$d_i d_j = d_{j-1} d_i \qquad \text{if } \ i < j,$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } \ i < j, \\ id & \text{if } i = j \ \text{ or } j+1, \\ s_j d_{i-1} & \text{if } i > j+1, \end{cases}$$

$$s_i s_j = s_j s_{i-1} \qquad \text{if } \ i > j.$$

Generally this structure is abstracted to give a family of sets, $\{K_n : n \geq 0\}$, face maps $d_i : K_n \to K_{n-1}$ and degeneracy maps, $s_i : K_n \to K_{n+1}$, satisfying these simplicial identities. The result is a *simplicial set*.

Remark: Using the singular complex, we can proceed much as in our earlier discussion to define *singular homology groups* for a space. Starting from $Sing(X)$, take a free Abelian group in each dimension then take the alternating sum of the faces to get a boundary map and thus a chain complex, $C(X)$, then take the homology of that. (We do not give details as this is very readily available in standard texts on algebraic topology.)

If $\mathcal{C}$ is any category, a simplicial object in $\mathcal{C}$ is given by a family of objects of $\mathcal{C}$, $\{K_n : n \geq 0\}$ and *morphisms* d_i and s_i as above. If $\mathbf{\Delta}$ denotes the category of finite ordinal sets, $[n] = \{0 < 1 < \ldots < n\}$ and order preserving functions between them, then a simplicial object in $\mathcal{C}$ is simply a functor, $K : \mathbf{\Delta}^{op} \to \mathcal{C}$, so the obvious definition of a simplicial map will be a natural transformation of functors, $f : K \to L$. This translates as a family of morphisms, $f_n : K_n \to L_n$, compatible in the obvious way with the d_i and s_i.

We denote the category of simplicial objects in $\mathcal{C}$ by $Simp(\mathcal{C})$ or $Simp.\mathcal{C}$, but will shorten $Simp(Sets)$ to $\mathcal{S}$.

The category, $\mathcal{S}$, models all homotopy types of spaces. It is a presheaf category, so is a topos and has a lot of nice structure including products, and mapping space objects $\underline{\mathcal{S}}(K, L)$, where

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L).$$

Here $\Delta[n] = \mathbf{\Delta}(-, [n])$, the standard simplicial n-simplex. This has a special n-simplex, namely the element ι_n in $\Delta[n]_n$ determined by the identity map.

The Yoneda lemma, from category theory, gives us an isomorphism $\mathcal{S}(\Delta[n], K) \cong K_n$, and so, for any n-simplex, x, gives us a simplicial map $\ulcorner x \urcorner : \Delta[n] \to K$, which is sometimes called the *name*, or *representing map* of x. From $\ulcorner x \urcorner$, you get x back by evaluating on $\ulcorner x \urcorner$ on ι_n.

Examples of simplicial sets.

First let us have a trivial example, ..., trivial but often very useful.

Definition: Given a set, X, the discrete simplicial set, $K(X,0)$, is defined to have $K(X,0)_n = X$ for all n and to have all face and degeneracy maps given by the identity function on X. A simplicial set K is said to be *discrete* if it is isomorphic to one of form $K(X,0)$ for some set X. (An easy extension gives the notion of discrete simplicial object in a category.)

With more substance, we have the following examples:

(i) If $\mathcal{A}$ is a small category or a groupoid, we can form a simplicial set, $Ner(\mathcal{A})$, defined by $Ner(\mathcal{A})_n = Cat([n], \mathcal{A})$, with the obvious face and degeneracy maps induced by composition with the analogues of the δ_i and σ_i. The simplicial set, $Ner(\mathcal{A})$, is called the *nerve of the category* $\mathcal{A}$. An n-simplex in $Ner(\mathcal{A})$ is a sequence of n composable arrows in $\mathcal{A}$.

This is easier to understand in pictures:

$Ner(A)_0$ is the set of objects;

$Ner(A)_1$ is the set of arrows or morphisms;

$Ner(A)_2$ is the set of composable pairs of morphisms, so $\sigma \in Ner(\mathcal{A})_2$ will be of form $\sigma = (a_0 \overset{\alpha_1}{\to} a_1 \overset{\alpha_2}{\to} a_2)$. Visualising this as a triangle shows the faces more clearly:

$$\begin{tikzcd} & a_1 \arrow[dr, "\alpha_2"] & \\ a_0 \arrow[ur, "\alpha_1"] \arrow[rr, "\alpha_1\alpha_2"'] & & a_2 \end{tikzcd}$$

The case $Ner(A)_n$ for $n = 3$, etc. are left to you. This *is* worth doing if you have not seen it before.

Note that in these contexts, we will sometimes use composition in the 'left-to-right' order, but in general categorical settings will use gf being first do f then g. To stick exclusively to one or the other is usually awkward, so we use both as appropriate. This sometimes means we have to take extra care over the conventions that we are using at a particular time.

If we have a group, G, consider it as the one object groupoid $G[1]$ as before, then $Ner(G[1])$ is really the simplicial set corresponding to our construction of the bar resolution of G. It is called the *nerve of* G, and is a *classifying space* for G, an aspect that we will explore later in some detail.

If we have a *discrete category* $\mathcal{A}$, i.e. $\mathcal{A}$ has no non-identity morphisms between objects, then $\mathcal{A}$ is really just a set, and $Ner(\mathcal{A})$ is a discrete simplicial set.

(ii) Suppose we have a simplicial complex K, then it almost is a simplicial set. There are some problems, but they are easily resolved. If we, a bit naïvely, set K_n to be the set of n-simplices of K, then how are we to define the face maps, and if K has no simplices in dimensions greater than n say, K_{n+1} will be empty so degeneracies cause problems as you cannot map from a non-empty set to an empty one!

That was too naïve, so we pick a partial order on the vertices of K such that any simplex is totally ordered, (for instance, a total order on $V(K)$ does the job, but may not be convenient sometimes and so may be 'overkill'). Now, reset K_n to be the set of all ordered strings, $\sigma = \langle x_0, \ldots, x_n\rangle$ of vertices, for which the underlying (unordered) set is a simplex of K. The degeneracies now can be handled simply. For example, if $\sigma = \langle x_0, x_1\rangle$ is a 1-simplex in this simplicial set, then $s_0\sigma = \langle x_0, x_0, x_1\rangle$, whilst $s_1\sigma = \langle x_0, x_1, x_1\rangle$. (The details are left to you to complete. Note we did not specify how to define the face maps, so you need to do that as well and to verify that it all fits together neatly.)

If you want to learn more about simplicial set theory, the old paper of Curtis, [53] and Peter May's monograph, [97], are very readable. There is a fairly well behaved notion of homotopy in $\mathcal{S}$, and simplicial homotopy theory is the subject of many good books. A chatty introduction to it can be found in Kamps and Porter, [81], which, of course, is highly recommended!

The homotopy theory of simplicial sets yields a notion of weak equivalence. (This is similar to 'quasi-isomorphism' in the homotopy theory of chain complexes.) There are homotopy groups and $f : K \to L$ is a weak equivalence if f induces isomorphisms on all homotopy groups. We will not need the detailed definition yet.

We next look at some simplicial algebraic gadgets, especially simplicial groups and simplicially enriched groupoids. We will concentrate on the first but must mention the second for completeness.

1.3.2 Simplicial Objects in Categories other than *Sets*

If $\mathcal{A}$ is any category, we can form $Simp.\mathcal{A} = \mathcal{A}^{\mathbf{\Delta}^{op}}$. (Sometimes we will use a variant notation: $Simp(\mathcal{A})$, as occasionally the first notation may be ambiguous.)

These categories often have a good notion of homotopy as briefly mentioned above; see also the discussion of simplicially enriched categories in [81]. Of particular use are:

(i) *Simp.Ab*, the category of simplicial Abelian groups. This is equivalent to the category of chain complexes by the Dold-Kan theorem, which we will mention in more detail later.

(ii) *Simp.Grps*, the category of simplicial groups. This 'models' all connected homotopy types, by Kan, [83] (cf., Curtis, [53]). There are adjoint functors $G : \mathcal{S}_{conn} \to Simp.Grps$, $\overline{W} : Simp.Grps \to \mathcal{S}_{conn}$, with the two natural maps $G\overline{W} \to Id$ and $Id \to \overline{W}G$ being weak equivalences.

Results on simplicial groups by Carrasco, [37], generalise the Dold-Kan theorem to the non-Abelian case, (cf., Carrasco and Cegarra, [38]).

(iii) '*Simp.Grpds*': in 1984 Dwyer and Kan, [59], (and also Joyal and Tierney, and Duskin and van Osdol, cf., Nan Tie, [103, 104]) noted how to generalise the $(G, \overline{W})$ adjoint pair to handle all simplicial sets, not just the connected ones. (Beware there are several important printing errors in the paper [59].) For this they used a special type of simplicial groupoid. Although the term used in [59] was exactly that, 'simplicial groupoid', this is really a misnomer and may give the wrong impression, as not all simplicial objects in the category of groupoids are used. A probably better term would be 'simplicially enriched groupoid', although 'simplicial groupoid with discrete objects' is also used. We will denote this category by $\mathcal{S}-Grpds$.

This category 'models' all homotopy types using a mix of algebra and combinatorial structure.

We will later describe both G and $\overline{W}$ in some detail, and will use simplicially enriched groupoids and simplicially enriched categories as well.

(iv) *Nerves of internal categories:* Suppose that $\mathcal{D}$ is a category with finite limits and C is an internal category in $\mathcal{D}$. What does that mean? In our earlier discussion on groupoids, we had the diagram that looked a bit like

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & C_0 \arrow[l, shift left=2, "i"] \end{tikzcd} .$$

We complete this one stage to build in the set of composable pairs $C_2 = C_1 \times_{C_0} C_1$ and the multiplication/ composition map, which we denote here by m.

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] & C_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"] & C_0 \arrow[l, shift left=2, "i"] \end{tikzcd}.$$

We did this previously within the category of sets, but could do it equally well in $\mathcal{D}$. We should also mention an object C_3 given by a 'triple pullback', which is useful when discussing the associativity of composition. This will give us the analogue of a small category, but in which the object of objects and the object of arrows are both themselves objects of $\mathcal{D}$ and the source target and composition maps are all morphisms in that category.

If one interprets this for $\mathcal{D} = Sets$, it becomes clear that this diagram that we seem to be building is part of the diagram specifying the nerve of the small category, C, with C_0 the set of objects, C_1 that of morphisms, C_2 that of composable pairs and so on. (We have not specified the two degeneracies from C_1 to C_2 in the diagram, but this is merely because we left the details of the rules governing identities out of our earlier discussion.) This builds a simplicial object in $\mathcal{D}$ as follows: take an n-fold pullback to get

$$C_n = \underbrace{C_1 \times_{C_0} C_1 \times_{C_0} C_1 \times_{C_0} \ldots \times_{C_0} C_1}_{n},$$

define face and degeneracies by the same sort of rules as in the set based nerve, that is, in dimension n, d_0 and d_n each leave out an end, whilst the d_i use the composition in the category to get a composite of two adjacent 'arrows', and the degeneracies are 'insertion of identities'. (Working out how to do these morphisms in terms of diagrams is quite fun!) We thus get a simplicial object in $\mathcal{D}$ called the *nerve of the internal category, C*. We will use this in several situations later in a key way. In particular, we will use the case $\mathcal{D} = Grps$.

Later on, we will use internal functors and natural transformations as well. For the moment, the description of these structures is **left to you**. Notationally, we will write $Cat(\mathcal{D})$ for the category of internal categories in $\mathcal{D}$. As you might expect, the above nerve construction is a functor from $Cat(\mathcal{D})$ to $Simp(\mathcal{D})$. (If you know about such things, you might also expect that $Cat(\mathcal{D})$ can be thought of as a 2-category, ..., you would be right, but we will leave that until much later on.)

(v) *Bisimplicial and multisimplicial objects:* A useful category in which we can take simplicial objects is $\mathcal{S}$ itself, and the same is true for other categories of form $Simp(\mathcal{A})$. For simplicity we will start by looking at simplicial objects in $\mathcal{S}$.

As a simplicial object in a category $\mathcal{A}$ is just a functor from $\mathbf{\Delta}^{op}$ to $\mathcal{A}$, a simplicial object in $\mathcal{S}$ is such a functor taking values that themselves are functors from $\mathbf{\Delta}^{op}$ to $Sets$. Another way to look at these is a 'functor of two variables' using a categorical version of the way that a function of two variables, $f : X \times Y \to Z$, can be thought of as a function $\tilde{f} : X \to Z^Y$ from X to the set of functions from Y to Z. Of course, $f(x, y) = \tilde{f}(x)(y)$ and similarly for the functors. We thus have a description of a simplicial object in $\mathcal{S}$ as corresponding to a functor $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to Sets$.

Definition: A *bisimplicial set* is a functor $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to Sets$. . A *morphism of bisimplicial sets*, $f : X \to Y$ is a natural transformation between the corresponding functors. More generally a *bisimplicial object in a category* $\mathcal{A}$ is a functor $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathcal{A}$, similarly for the corresponding morphisms. The corresponding categories will denoted $BiS := BiSimp(Sets)$ and in general $BiSimp(\mathcal{A})$.

A simplicial set can be specified by giving sets X_n and face and degeneracy 'operators' between them satisfying the simplicial idenities. A bisimplicial set is similarly specified by a bi-indexed family of sets $X_{p,q}$ and two families of simplicial operators. We may use the terms 'horizontal' and 'vertical' for these two families as that is how the corresponding diagrams are often drawn. For instance, the bottom part of a bisimplicial set will look a bit like the following:

$$\begin{tikzcd}
 & \vdots \arrow[d, shift left=2, "d_0^v"] \arrow[d] \arrow[d, shift right=2, "d_2^v"'] & \vdots \arrow[d, shift left=2, "d_0^v"] \arrow[d] \arrow[d, shift right=2, "d_2^v"'] \\
\cdots \arrow[r, shift left=2, "d_0^h"] \arrow[r] \arrow[r, shift right=2, "d_2^h"'] & X_{1,1} \arrow[r, shift left, "d_0^h"] \arrow[r, shift right, "d_1^h"'] \arrow[d, shift left, "d_0^v"] \arrow[d, shift right, "d_1^v"'] & X_{0,1} \arrow[d, shift left, "d_0^v"] \arrow[d, shift right, "d_1^v"'] \\
\cdots \arrow[r, shift left=2, "d_0^h"] \arrow[r] \arrow[r, shift right=2, "d_2^h"'] & X_{1,0} \arrow[r, shift left, "d_0^h"] \arrow[r, shift right, "d_1^h"'] & X_{0,0}
\end{tikzcd}$$

(As usual in such diagrams, there is not really room to show the degeneracy maps and so these are omitted from the picture.) In addition to the simplicial identities holding in each direction, each horizontal face or degeneracy has to be a simplicial map between the vertical simplicial sets. Practically this means that the diagram must commute.

We will later meet bisimplicial groups, and also briefly multisimplicial objects in which the number of variables is not limited to two. For instance, the nerve of a simplicial group is most naturally viewed as a bisimplicial set, and similarly the nerve of a bisimplicial group is a trisimplicial set, that is a functor from $\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ to $Sets$. There are ways of passing between such things as we will see later.

(vi) *Cosimplicial things:* At certain points in the development of cohomology and related areas we will have need to talk of cosimplicial sets.

Definition: A *cosimplicial set* is a functor $K : \mathbf{\Delta} \to Sets$, and a morphism of such is a natural transformation between the corresponding functors. The category of such will be denoted $CoSimp(Sets)$, and similarly for the obvious generalisations to other settings, namely *cosimplicial objects in a category* $\mathcal{A}$, being functors $K : \mathbf{\Delta} \to \mathcal{A}$ with corresponding morphisms forming a category $CoSimp(\mathcal{A})$.

This looks at one and the same time very similar and very different to simplicial objects. Certainly analysis of, say, simplicial groups is much easier than that of cosimplicial groups, but, as any functor, $K : \mathbf{\Delta} \to \mathcal{A}$, gives uniquely a functor, $K^{op} : \mathbf{\Delta}^{op} \to \mathcal{A}^{op}$, a cosimplicial object is also a simplicial object in the opposite category. The problem, thus, is that often the opposite category of a well known category, such as that of groups, is a lot less nice. Even the dual of $Sets$ is not that 'well behaved'.

Conjugation: There is an 'inversion' operation on each finite ordinal in $\mathbf{\Delta}$, which forms reverse the order on the ordinal, that is, it sends $\{0 < 1 < \ldots < n\}$ to $\{0 > 1 > \ldots > n\}$. Of course the resulting object is isomorphic to the original, but is not compatible with the face or degeneracy maps. This operation induces an operation on simplicial objects, that we will call *conjugation.*

Definition: Given a simplicial object, X in a category $\mathcal{A}$, the *conjugate simplicial object*, $ConjX$, is defined by

$$(ConjX)_n = X_n,$$

$$d_i : (ConjX)_n \to (ConjX)_{n-1} = d_{n-i} : X_n \to X_{n-1}$$

for each $0 \leq i \leq n$, and, similarly,

$$s_i : (ConjX)_n \to (ConjX)_{n+1} = s_{n-i} : X_n \to X_{n+1}.$$

Clearly X and $ConjX$ are closely related. For instance, they have isomorphic geometric realisation, isomorphic homotopy groups, ..., but the actual comparisons are quite difficult to give because there are, in general, very few simplicial morphisms from X to $ConjX$.

Example: In some contexts, a situation naturally leads to a variant form of the nerve functor being used. Suppose that $\mathcal{A}$ is a category. Our usual notation for an n-simplex in $Ner(\mathcal{A}$ would be something like $(a_0 \stackrel{\alpha_1}{\to} a_1 \to \ldots \stackrel{\alpha_n}{\to} a_n)$, but sometimes the order of the terms is reversed as it is more natural, in certain situations, to use $(a'_n \stackrel{\alpha'_n}{\to} a'_{n-1} \to \stackrel{\alpha'_1}{\to} a'_0)$. This might typically arise if one has a right action of some group instead of the left actions that we will tend to meet more often. It also occurs sometimes in the way that terms of the Bousfield-Kan form of the homotopy colimit construction are presented, (see the comment on page **??**). The link between the two forms is $a'_i = a_{n-i}$ and $\alpha'_i = \alpha_{n-i+1}$. The face operators delete or compose in the conjugate way. Of course, the nerve based on this notational form is the conjugate of the one we have defined earlier. We will refer to it as the *conjugate nerve* of the category.

1.3.3 The Moore complex and the homotopy groups of a simplicial group

Given a simplicial group G, the *Moore complex*, (NG, ∂), *of* G is the chain complex defined by

$$NG_n = \bigcap_{i=1}^{n} Ker\, d_i^n$$

with $\partial_n : NG_n \to NG_{n-1}$ induced from d_0^n by restriction. (Note there is no assumption that the NG_n are Abelian.)

The n^{th} *homotopy group*, π_n(G), of G is the n^{th} homology of the Moore complex of G, i.e.,

$$\begin{aligned}\pi_n(G) &\cong H_n(NG, \partial),\\ &= \big(\textstyle\bigcap_{i=0}^{n} Ker\, d_i^n\big)/d_0^{n+1}\big(\textstyle\bigcap_{i=1}^{n+1} Ker\, d_i^{n+1}\big).\end{aligned}$$

(You should check that $\partial NG_{n+1} \triangleleft NG_n$.)

The interpretation of NG and $\pi_n(G)$ is as follows:

for $n = 1$, $g \in NG_1$,

$$1\bullet \xrightarrow{g} \bullet\partial g$$

and $g \in NG_2$ looks like

$$\begin{tikzcd}
& \bullet \arrow[dr, "\partial g"] \arrow[d, phantom, "g"] & \\
\bullet \arrow[ur, "1"] \arrow[rr, "1"'] & & \bullet
\end{tikzcd}$$

and so on.
We note that $g \in NG_2$ is in $Ker\,\partial$ if it looks like

$$\begin{tikzcd}
& \bullet \arrow[dr, "1"] \arrow[d, phantom, "g"] & \\
\bullet \arrow[ur, "1"] \arrow[rr, "1"'] & & \bullet
\end{tikzcd}$$

whilst it will give the trivial element of $\pi_2(G)$ if there is a 3-simplex x with g on its third face and all other faces identity.

This simple interpretation of the elements of NG and $\pi_n(G)$ will 'pay off' later by aiding interpretation of some of the elements in other situations. The homotopy groups we have introduced above have been defined purely algebraically as homology of a related complex. Any simplicial group gives us a base pointed simplicial set simply by forgetting the group structure and taking the identity element as the base point. Any pointed simplicial set gives homotopy groups in two different ways. There is an intrinsic way that is described in detail in, for instance, May's book, [97], but they can also be defined via a geometric realisation, which produces a space from the simplicial set. These two ways always give the same answer, and in the case that we are looking at of an underlying simplicial set of a simplicial group, this group coincides with that defined via the Moore complex. (This is easily found in the literature if you want to check up on it, so we will not repeat it here.)

n-equivalences and homotopy n-types Let $n \geq 0$. A morphism, $f : G \to H$, of simplicial group(oid)s is an *n-equivalence* if the induced homomorphisms, $\pi_k(f) : \pi_k(G) \to \pi_k(H)$ are isomorphisms for all $k < n$.

Inverting the n-equivalences in *Simp.Grps* gives a category Ho_n(*Simp.Grps*) and two simplicial groups have the same *n-type* if they are isomorphic in Ho_n(*Simp.Grps*).

Remark and warning: For a space or simplicial set K, $\pi_k(K) \cong \pi_{k-1}(\mathcal{G}(K))$, so these simplicial group n-types correspond to restrictions on $\pi_k(K)$ for $k \leq n$ in the spatial context.

To consider the application of this to homotopical and homological algebra, we will also need the following:

Definitions: (i) A simplicial group, G, is *augmented* by specifying a constant simplicial group $K(G_{-1}, 0)$ and a surjective group homomorphism, $f = d_0^0 : G_0 \to G_{-1}$ with $fd_0^1 = fd_1^1 : G_1 \to G_{-1}$. An *augmentation* of the simplicial group G is then a map

$$G \longrightarrow K(G_{-1}, 0),$$

where $K(G_{-1}, 0)$ is the constant simplicial group with value G_{-1}.

(ii) An augmented simplicial group, (G, f), is *acyclic* if the corresponding complex is acyclic, i.e., $H_n(NG) \cong 1$ for $n > 0$ and $H_0(NG) \cong G_{-1}$.

Remarks: (i) The above notions are just particular instances of the general notion of an *augmented simplicial object* in a category, and the corresponding idea of *acyclic* such things in settings where the definition makes sense.

(ii) When considering augmented simplicial objects, we sometimes use the notation d_0 or d_0^0 for the augmentation map as then the condition $fd_0^1 = fd_1^1$ becomes $d_0d_0 = d_0d_1$, which is a natural extension of the simplicial identities.

1.3.4 Kan complexes and Kan fibrations

Within the category of simplicial sets, there is an important subcategory determined by those objects that satisfy the Kan condition, that is the *Kan complexes*.

As before we set $\Delta[n] = \mathbf{\Delta}(-, [n]) \in \mathcal{S}$, then, for each i, $0 \leq i \leq n$, we can form, within $\Delta[n]$, a subsimplicial set, $\Lambda^i[n]$, called the *(n, i)-horn* or *(n, i)-box*, by discarding the top dimensional n-simplex (given by the identity map on $[n]$) and its i^{th} face. We must also discard all the degeneracies of those simplices.

By an *(n, i)-horn* or *box* in a simplicial set K, we mean a simplicial map $f : \Lambda^i[n] \to K$. Such a simplicial map corresponds intuitively to a family of n simplices of dimension $(n-1)$, fitting together to form a 'funnel' or 'empty horn' shaped subcomplex within K. The family is thus a sequence, $(k_0, \ldots, k_{i-1}, -, k_{i+1}, \ldots, k_n)$, with each $k_\ell \in K_{n-1}$, satisfying $d_\ell k_j = d_{j-1}k_\ell$, for $\ell < j$, whenever both k_ℓ and k_j are in the sequence. The idea is that a Kan fibration of simplicial sets is a map in which the horns in the domain can be 'filled' if their images in the codomain can be. More formally:

Definition: A map $p : E \to B$ is a *Kan fibration* if, for any n, i as above, given any (n, i)-horn in E, specified by a map $f_1 : \Lambda^i[n] \to E$, together with an n-simplex, $f_0 : \Delta[n] \to B$, such that

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] & B \end{tikzcd}$$

commutes, then there is an $f : \Delta[n] \to E$ such that $pf = f_0$ and $f.inc = f_1$, i.e., f lifts f_0 and extends f_1.

We also say that p satisfies the *Kan lifting condition* if this is true.

Definition: A simplicial set, K, is a *Kan complex* if the unique map $K \to \Delta[0]$ is a Kan fibration. This is equivalent to saying that every horn in K has a filler, i.e., any $f_1 : \Lambda^i[n] \to Y$ extends to an $f : \Delta[n] \to Y$.

Singular complexes, $Sing(X)$, and the simplicial mapping spaces, $\underline{Top}(X, Y)$, are always Kan complexes.

Lemma 1 *The nerve of a category, $\mathcal{C}$, is a Kan complex if and only if the category is a groupoid.* ■

The proof is left to **the reader**.

This is very important as the filler structure involves compositions and inverses, so encodes the *algebraic* structure of $\mathcal{C}$. Later we will use this many times, sometimes explicitly, but often it will be giving structure behind the scenes, for instance, internally within some other category.

There is a property of Kan fibrations, that is *very useful*, namely that *the pullback of a Kan fibration along a simplicial map is again a Kan fibration.* More precisely:

Proposition 1 *Let $p : E \to B$ be a Kan fibration, and let $f : X \to B$ be a simplicial map, and form the pullback of p along f, written $f^*(p) : E_f \to X$. This map is a Kan fibration.*

Proof: (Just to help you think about $f^*(p) : E_f \to X$ more concretely, first note that $f^*(p) : E_f \to X$ is only really defined up to isomorphism as it is given by a universal property in the usual way, but we can find a particular 'model' of that isomorphism class of potential things as follows. Look at the simplicial set $X \times_B E$, where

$$(X \times_B E)_n = \{(x, e) \mid x \in X_n, e \in E_n, f(x) = p(e)\}$$

and where face and degeneracy maps are defined componentwise, so $d_i(x, e) = (d_i(x), d_i(e))$, etc. The map, $f^*(p)$ is then represented by the first projection. We will not use this model explicitly. It is just there to help you if need be. Make sure you have looked up the universal property of pullbacks as we will need it.)

We have a pullback square:

$$\begin{tikzcd} E_f \arrow[r, "f'"] \arrow[d, "f^*(p)"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

Now assume we are given a diagram

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E_f \arrow[d, "f^*(p)"] \\ \Delta[n] \arrow[r, "f_0"'] & X \end{tikzcd}$$

and we seek a lift of f_0 to E_f. Composing f_0 and f on the base, and f_1 and f' up top, and using the Kan fibration property of p, we get a lift, g, of ff_0 to E. (**Draw the diagram.**) Using the maps f_0 and g, you check that $ff_0 = pg$, and the universal property of the original pullback square gives you a map, h, say, to E_f. It now just remains to check that this is a lift of f_0, and an extension of f_1, and checking that is **left to you**. ■

This result is often stated by saying that the class of Kan fibrations is *pullback stable.*

1.3.5 Simplicial groups are Kan

If G is a simplicial group, then its *underlying simplicial set* is a Kan complex. Moreover, given a box in G, there is an algorithm for filling it using products of degeneracy elements. A form of this algorithm is given below. More generally if $f : G \to H$ is an epimorphism of simplicial groups, then the underlying map of simplicial sets is a Kan fibration.

The following description of the algorithm is adapted from May's monograph, [97], page 67.

Proposition 2 *Let G be a simplicial group, then every box has a filler.*

Proof: Let $(y_0, \ldots, y_{k-1}, -, y_{k+1}, \ldots, y_n)$ give a horn in G_{n-1}, so the y_is are $(n-1)$ simplices that fit together as if they were all but one, the k^{th} one, of the faces of an n-simplex. There are three cases:

(i) $k = 0$: Let $w_n = s_{n-1}y_n$ and then $w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i$ for $i = n, \ldots, 1$, then w_1 satisfies $d_iw_1 = y_i$, $i \neq 0$;

(ii) $0 < k < n$: Let $w_0 = s_0y_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i$ for $i = 0, \ldots, k-1$, then take $w_n = w_{k-1}(s_{n-1}d_nw_{k-1})^{-1}s_{n-1}y_n$, and finally a downwards induction given by $w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i$, for $i = n, \ldots, k+1$, then w_{k+1} gives $d_iw_{k+1} = y_i$ for $i \neq k$;

(iii) the third case, $k = n$ uses $w_0 = s_0y_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i$ for $i = 0, \ldots, n-1$, then w_{n-1} satisfies $d_iw_{n-1} = y_i$, $i \neq n$. ∎

Some discussion of how you can think of this algorithm can be found in [81].

(You could see if you can adapt the idea of this proof to prove the result mentioned immediately before the statement, namely: if $f : G \to H$ is an epimorphism of simplicial groups, then the underlying map of simplicial sets is a Kan fibration. What about the converse?)

Later on we will meet the simplicial mapping space, $\underline{\mathcal{S}}(K, L)$, of simplicial maps from K to L. It is defined by $\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L)$, with the obvious induced maps. It is easy to see that if L is a Kan complex, then so is $\underline{\mathcal{S}}(K, L)$, for any K. (**Try to prove it**, but then look at May, [97], to compare your attempt with his proof.) This result has a useful generalisation that we will state as a lemma, but again will leave **you to give or find a proof**.

Lemma 2 *If $p : L \to M$ is a Kan fibration, and K is an arbitrary simplicial set, then the induced map, $\underline{\mathcal{S}}(K, p) : \underline{\mathcal{S}}(K, L) \to \underline{\mathcal{S}}(K, M)$, is also one.* ∎

(To give you a hint consider what a horn in $\underline{\mathcal{S}}(K, L)$ looks like, and likewise what an n-simplex in $\underline{\mathcal{S}}(K, M)$ is. Why should you be able to put the information together to build an n-simplex in $\underline{\mathcal{S}}(K, L)$? Look at low dimensional examples to build up some geometric intuition about what is going on. That is important even if you later look up a proof as not every proof that you will find gives the intuitive idea behind.)

1.3.6 T-complexes

There is quite a difference between the Kan complex structure of the nerve of a groupoid, G, and that of a singular complex. In the first, if we are given a (n, i)-horn, then there is *exactly one* n-simplex in $Ner(G)$, since the (n, i)-horn has a chain of n-composable arrows of G in it (at least unless $(n, i) = (2, 0)$ or $(2, 2)$, which cases are **left to you**) and that chain gives the required

n-simplex. In other words, there is a 'canonical' filler for any horn. In $Sing(X)$, there will usually be many fillers. (Think about why this is true.)

One attempt to handle 'canonical fillers' interacts with a notion that we will encounter later on, namely that of crossed complexes, for which see section 3.1. The resulting notion of a simplicial T-complex is one sort of 'Kan complex with canonical fillers' and various of the intuitions and arguments that this introduces will recur frequently in the following chapters. It assumes there is always a unique special filler. There may be other non-special ones, but that is not controlled in the process, as we will see. Simplicial T-complexes were introduced by Dakin, [54]:

Definition: A *simplicial T-complex* consists of a pair (K, T), where K is a simplicial set and $T = (T_n)_{n\geq 1}$ is a graded subset of K with $T_n \subseteq K_n$. Elements of T are called *thin*. The thin structure satisfies the following axioms:

$T.1$ Every degenerate element is thin.

$T.2$ Every box in K has a unique thin filler.

$T.3$ A thin filler of a thin box also has its last face thin.

Example: The nerve of a groupoid has a T-complex structure in which each simplex of dimension greater than or equal to 2 is thin. Our earlier comments give the proof. Conversely, if (K, T) is a T-complex with $T_n = K_n$ for all $n \geq 2$, then K is the nerve of a groupoid with set of objects K_0 and set of arrows, K_1. (It is **left to you** to see how to compose arrows, to prove that it is an associative composition, and that there are identities at all objects.)

A box or horn is, of course, as in section 1.3.4, a collection of n-simplices that fits together like the collection of all but one faces of an $(n+1)$-simplex. The collection of such n-boxes with given face missing can be formulated in terms of a pullback and hence axioms $T2$ and $T3$ can be encoded in a form suitable for adapting to other contexts. Similar ideas are used by Duskin, [56], and Nan-Tie, [103, 104], and we will have occasion to refer back to these later. We will need to adapt those ideas initially to T-complexes within the setting of groups (group T-complexes as below) but later we may need them in various other settings. Group T-complexes were briefly considered by Ashley, [6], but their main theory has been clarified and extended by Carrasco, [37], and Cegarra and Carrasco, [38], using ideas that will be discussed briefly later.

1.3.7 Group T-complexes

Definition: A *group T-complex* is "a T-complex (G, T) in which G is a simplicial group and T is a graded subgroup of G", (Ashley, [6]).

Ashley proved a series of results that gave a neat alternative formulation of this concept. We note the following observations:

Lemma 3 *Let $D = (D_n)_{n\geq 1}$ be the graded subgroup of G generated by the images of the degeneracy maps, $s_i : G_n \to G_{n+1}$, for all i and n, then any box in G has a standard filler in D.*

Proof: In fact, the algorithmic formulae used when proving that any simplicial group is a Kan complex (cf., Proposition 2) give a filler defined as a product of degenerate copies of the faces of the box. ■

Proposition 3 *If (G, T) is a group T-complex then $T = D$.*

Proof: To see this, we note that axiom $T1$ implies that $D \subseteq T$. Conversely if $t \in T_n$, then it fills the box made up of $(\_, d_1 t, \ldots, d_n t)$. This, in turn, has a filler, d, in D, but, as this filler is also thin, it must be that $t = d$, since thin fillers are uniquely determined ($T2$). ■

This is neat since it says there is essentially at most one group T-complex structure on any given simplicial group. The next results says when such a structure does exist.

Theorem 1 *(Ashley, [6]) If G is a simplicial group, then (G, D) is a group T-complex if and only if $NG \cap D$ is the trivial graded subgroup.*

Proof: One way around, this is nearly trivial. If (G, D) is a group T-complex and $x \in NG_n$, then x fills a box $(\_, 1, \ldots, 1)$, so if $x \in NG_n \cap D_n$, x must itself be the thin filler, however 1 is also a thin filler for this box, so $x = 1$ as required.

Conversely if $NG \cap D = \{1\}$, then we must check $T2$ and $T3$, $T1$ being trivial. As any box has a standard filler in D, we only have to check uniqueness, but if x and y are in D_n, and both fill the same box (with the k^{th} face missing) then $z = xy^{-1}$ fills a box with 1s on all faces (and the k^{th} face missing).

If $k = 0$, then as $z \in NG_n \cap D_n$, we have $z = 1$ and x and y are equal. If $k > 0$, assume that if $\ell < k$ and $z \in D_n \cap \bigcap_{i \neq \ell} Ker\, d_i$ then $z = 1$, (i.e, that we have uniqueness up to at least the $(k-1)^{st}$ case). Consider $w = z s_{k-1} d_k z^{-1}$. This is still in D_n and $d_i w = 1$ unless $i = k-1$, hence by assumption $w = 1$. Of course, this implies that $z = s_{k-1} d_k z$, but then $d_{k-1} z = d_k z$. We know that $d_{k-1} z = 1$, so $d_k z = 1$ and $z = 1$, i.e., $x = y$ and we have uniqueness at the next stage.

To verify $T3$, assume that $x \in D_{n+1}$ and each $d_i x \in D_n$ for $i \neq k$, then we can assume that $k = 0$, since otherwise we can skew the situation around as before to get that to be true, verify it in that case and 'skew' it back again later. Suppose therefore that $d_i x \in D_n$ for all $0 < i < n$. As x must be the degenerate filler given by the standard method, we can calculate x as follows: let $w_n = s_{n-1} d_n x$, $w_i = w_{i+1} (s_{i-1} d_i w_{i+1})^{-1} s_{i-1} y_i$ for $i = 1$, then $x = w_1$. We can therefore check that $d_0 x \in D_n$ as required. ■

Remark: Ashley, [6], in fact assumes a seemingly stronger conclusion, namely that $D_n \cap \bigcup_{\ell=0}^{n} (\bigcap_{i \neq \ell} Ker\, d_i) = 1$. The reduction to the single case is noted by Carrasco, [37].

Thus a group T-complex is a simplicial group in which the Moore complex contains no non-trivial product of degenerate elements.

It is often useful to have a 'dimensionwise' terminology in the following sense. We could say that a group T-complex satisfies the *thin filler condition* or simply, the *T-condition*, in all dimensions. That suggests that we extract that condition 'dimensionwise' as follows:

Definition: A simplicial group G satisfies the *thin filler condition in dimension n* if $NG_n \cap D_n$ is trivial. We may abbreviate that to *T-condition in dimension n*.

This terminology lends itself well to such variants as 'G satisfies the *thin filler condition in dimensions greater that* k' meaning that $NG_n \cap D_n$ is trivial for all $n > k$, and so on.

It is left as an exercise to prove that any simplicial Abelian group is a group T-complex. (At this stage, this is moderately challenging, and it may help to take a brief look at the later section on Conduché's decomposition and the Dold-Kan theorem.)

Chapter 2

Crossed modules - definitions, examples and applications

We will give these for groups, although there are analogues for many other algebraic settings.

2.1 Crossed modules

Definition: A *crossed module*, (C, G, δ), consists of groups C and G with a left action of G on C, written $(g, c) \to {}^{g}c$ for $g \in G$, $c \in C$, and a group homomorphism $\delta : C \to G$ satisfying the following conditions:
CM1) for all $c \in C$ and $g \in G$,

$$\delta({}^{g}c) = g\delta(c)g^{-1},$$

CM2) for all $c_1, c_2 \in C$,

$${}^{\delta(c_2)}c_1 = c_2 c_1 c_2^{-1}.$$

(CM2 is called the *Peiffer identity.*)

If (C, G, δ) and (C', G', δ') are crossed modules, a *morphism*, $(\mu, \eta) : (C, G, \delta) \to (C', G', \delta')$, *of crossed modules* consists of group homomorphisms $\mu : C \to C'$ and $\eta : G \to G'$ such that
(i) $\delta'\mu = \eta\delta$ and (ii) $\mu({}^{g}c) = {}^{\eta(g)}\mu(c)$ for all $c \in C$, $g \in G$.

Crossed modules and their morphisms form a category, of course. It will usually be denoted $CMod$.

There is, for a fixed group G, a subcategory $CMod_G$ of $CMod$, which has, as objects, those crossed modules with G as the "base", i.e., all (C, G, δ) for this fixed G, and having as morphisms from (C, G, δ) to (C', G, δ') just those (μ, η) in $CMod$ in which $\eta : G \to G$ is the identity homomorphism on G.

Several well known situations give rise to crossed modules. The verification will be left to you.

2.1.1 Algebraic examples of crossed modules

(i) Let H be a normal subgroup of a group G with $i : H \to G$ the inclusion, then we will say (H, G, i) is a *normal subgroup pair.* In this case, of course, G acts on the left of H by conjugation and the inclusion homomorphism i makes (H, G, i) into a crossed module, an 'inclusion crossed modules'. Conversely it is an easy exercise to prove

Lemma 4 *If (C, G, ∂) is a crossed module, ∂C is a normal subgroup of G.* ∎

(ii) Suppose G is a group and M is a left G-module; let $0 : M \to G$ be the trivial map sending everything in M to the identity element of G, then $(M, G, 0)$ is a crossed module.

Again conversely:

Lemma 5 *If (C, G, ∂) is a crossed module, $K = Ker\,\partial$ is central in C and inherits a natural G-module structure from the G-action on C. Moreover, $N = \partial C$ acts trivially on K, so K has a natural G/N-module structure.* ∎

Again the proof is left as an exercise.

As these two examples suggest, general crossed modules lie between the two extremes of normal subgroups and modules, in some sense, just as groupoids lay between equivalence relations and G-sets. Their structure bears a certain resemblance to both - they are "external" normal subgroups, but also are "twisted" modules.

(iii) Let G be a group, then, as usual, let $Aut(G)$, denote the group of automorphisms of G. Conjugation gives a homomorphism

$$\iota : G \to Aut(G).$$

Of course, $Aut(G)$ acts on G in the obvious way and ι is a crossed module. We will need this later so will give it its own name, the *automorphism crossed module of the group, G* and its own notation: $\mathsf{Aut}(G)$.

More generally if L is some type of algebra then $U(L) \to Aut(L)$ will be a crossed module, where $U(L)$ denotes the units of L and the morphism send a unit to the automorphism given by conjugation by it.

This class of example has a very nice property with respect to general crossed modules. For a general crossed module, (C, P, ∂), we have an action of P on C, hence a morphism, $\alpha : P \to Aut(C)$, so that $\alpha(p)(c) = {}^{p}c$. There is clearly a square

$$\begin{tikzcd} C \arrow[r, "="] \arrow[d, "\partial"'] & C \arrow[d, "\iota"] \\ P \arrow[r, "\alpha"'] & Aut(C) \end{tikzcd}$$

and we can ask if this gives a morphism of crossed modules. 'Clearly' it should. The requirements are that the square commutes and that the actions are compatible in the obvious sense, (recall page 35). To see that the square commutes, we just note that, given $c \in C$, ∂c acts on an $x \in C$, by conjugation by c: ${}^{\partial c}x = c.x.c^{-1} = \iota(c)(x)$, whilst to check that the actions match correctly remember that $\alpha(p)(c) = {}^{p}x$ by definition, so we do have a morphism of crossed modules as expected.

(iv) We suppose given a morphism
$$\theta : M \to N$$
of left G-modules and form the semi-direct product $N \rtimes G$. This group we make act on M via the projection from $N \rtimes G$ to G.

We define a morphism
$$\partial : M \to N \rtimes G$$
by $\partial(m) = (\theta(m), 1)$, where 1 denotes the identity element of G, then $(M, N \rtimes G, \partial)$ is a crossed module. In particular, if A and B are Abelian groups, and B is considered to act trivially on A, then any homomorphism, $A \to B$ is a crossed module.

(v) Suppose that we have a crossed module, $\mathsf{C} = (C, G, \delta)$, and a group homomorphism $\varphi : H \to G$, then we can form the 'pullback group' $H \times_G C = \{(h, c) \mid \varphi(h) = \delta c\}$, which is a subgroup of the product $H \times C$. There is a group homomorphism, $\delta' : H \times_G C \to H$, namely the restriction of the first projection morphism of the product, (so $\delta'(h, c) = h$). **You are left** to construct an action of H on this group, $H \times_G C$ such that $\varphi^*(\mathsf{C}) := (H \times_G C, H, \delta')$ is a crossed module, and also such that the pair of maps φ and the second projection $H \times_G C \to C$ give a morphism of crossed modules.

Definition: The crossed module, $\varphi^*(\mathsf{C})$, thus defined, is called the *pullback crossed module of* C *along* φ

(vi) As a last algebraic example for the moment, let
$$1 \to K \xrightarrow{a} E \xrightarrow{b} G \to 1$$
be an extension of groups with K a central subgroup of E, i.e., a *central extension* of G by K. For each $g \in G$, pick an element $s(g) \in b^{-1}(g) \subseteq E$. Define an action of G on E by: if $x \in E$, $g \in G$, then
$$^g x = s(g)xs(g)^{-1}.$$
This is well defined, since if $s(g)$, $s'(g)$ are two choices, $s(g) = ks'(g)$ for some $k \in K$, and K is central. (This also shows that this *is* an action.) The structure (E, G, b) is a crossed module.

A particular important case is: for R a ring, let $E(R)$ be the group of elementary matrices of R, $E(R) \subseteq G\ell(R)$ and $St(R)$, the corresponding Steinberg group with $b : St(R) \to E(R)$, the natural morphism, (see later, page 99, or [99], for the definition). This, then, gives a central extension
$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$
and thus a crossed module. In fact, more generally,
$$b : St(R) \to G\ell(R)$$
is a crossed module. The group, $G\ell(R)/Im(b)$, is $K_1(R)$, the first algebraic K-group of the ring.

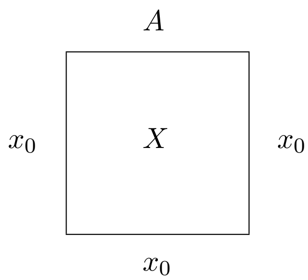
2.1.2 Topological Examples

In topology there are several examples that deserve looking at in detail as they do relate to aspects of the above algebraic cases. They require slightly more topological knowledge than has been assumed so far.

(vii) Let X be a pointed space, with $x_0 \in X$ as its base point, and A a subspace with $x_0 \in A$. Recall that the second relative homotopy group, $\pi_2(X, A, x_0)$, consists of relative homotopy classes of continuous maps

$$f : (I^2, \partial I^2, J) \to (X, A, x_0)$$

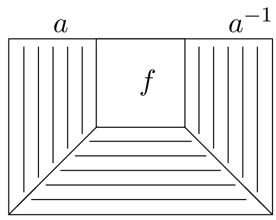
where ∂I^2 is the boundary of I^2, the square, $[0,1] \times [0,1]$, and $J = \{0,1\} \times [0,1] \cup [0,1] \times \{0\}$. Schematically f maps the square as:



so the top of the boundary goes to A, the rest to x_0 and the whole thing to X. The relative homotopies considered then deform the maps in such a way as to preserve such structure, so intermediate mappings also send J to x_0, etc. Restriction of such an f to the top of the boundary clearly gives a homomorphism

$$\partial : \pi_2(X, A, x_0) \to \pi_1(A, x_0)$$

to the fundamental group of A, based at x_0. There is also an action of $\pi_1(A, x_0)$ on $\pi_2(X, A, x_0)$ given by rescaling the 'square' given by



where f is partially 'enveloped' in a region on which the mapping is behaving like a.

Of course, this gives a crossed module

$$\pi_2(X, A, x_0) \to \pi_1(A, x_0).$$

A direct proof is quite easy to give. One can be found in Hilton's book, [73] or in Brown-Higgins-Sivera, [31]. Alternatively one can use the argument in the next example.

(viii) Suppose $F \stackrel{i}{\to} E \stackrel{p}{\to} B$ is a fibration sequence of pointed spaces. Thus p is a fibration, $F = p^{-1}(b_0)$, where b_0 is the basepoint of B. The fibre F is pointed at f_0, say, and f_0 is taken as the basepoint of E as well.

There is an induced map on fundamental groups

$$\pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(E)$$

and if a is a loop in E based at f_0, and b a loop in F based at f_0, then the composite path corresponding to aba^{-1} is homotopic to one wholly within F. To see this, note that $p(aba^{-1})$ is null homotopic. Pick a homotopy in B between it and the constant map, then lift that homotopy back up to E to one starting at aba^{-1}. This homotopy is the required one and its other end gives a well defined element ${}^ab \in \pi_1(F)$ (abusing notation by confusing paths and their homotopy classes). With this action $(\pi_1(F), \pi(E), \pi_1(i))$ is a crossed module. This will not be proved here, but is not that difficult. Links with previous examples are strong.

If we are in the context of the above example, consider the inclusion map, f of a subspace A into a space X (both pointed at $x_0 \in A \subset X$). Form the corresponding fibration,

$$i^f : M^f \to X,$$

by forming the pullback

$$\begin{tikzcd} M^f \arrow[r, "\pi^f"] \arrow[d, "j^f"'] & X^I \arrow[d, "e_0"] \\ A \arrow[r, "f"'] & X \end{tikzcd}$$

so M^f consists of pairs, (a, λ), where $a \in A$ and λ is a path from $f(a)$ to some point $\lambda(1)$. Set $i^f = e_1\pi^f$, so $i^f(a, \lambda) = \lambda(1)$. It is standard that i^f is a fibration and its fibre is the subspace $F_h(f) = \{(a, \lambda) \mid \lambda(1) = x_0\}$, often called the *homotopy fibre* of f. The base point of $F_h(f)$ is taken to be the constant path at x_0, (x_0, c_{x_0}).

If we note that

$$\pi_1(F_h(f)) \cong \pi_2(X, A, x_0)$$
$$\pi_1(M^f) \cong \pi_1(A, x_0)$$

(even down to the descriptions of the actions, etc.), the link with the previous example becomes clear, and thus furnishes another proof of the statement there.

(ix) The link between fibrations and crossed modules can also be seen in the category of simplicial groups. A morphism $f : G \to H$ of simplicial groups is a fibration if and only if each f_n is an epimorphism. This means that a fibration is determined by the fibre over the identity which is, of course, the kernel of f. The $(G, \overline{W})$-links between simplicial groups and simplicial sets mean that the analogue of π_1 is π_0. Thus the fibration f corresponds to

$$Ker\, f \stackrel{\lhd}{\to} G$$

and each level of this is a crossed module by our earlier observations. Taking π_0, it is easy to check that

$$\pi_0(Ker\, f) \to \pi_0(G)$$

is a crossed module. In fact any crossed module is isomorphic to one of this form. (Proof left to the reader.)

If $\mathsf{M} = (C, G, \partial)$ is a crossed module, then we sometimes write $\pi_0(\mathsf{M}) := G/\partial C$, $\pi_1(\mathsf{M}) := Ker\, \partial$, and then have a 4-term exact sequence:

$$0 \to \pi_1(\mathsf{M}) \to C \xrightarrow{\partial} G \to \pi_0(\mathsf{M}) \to 1.$$

In topological situations when M provides a model for (part of) the homotopy type of a space X or a pair (X, A), then typically $\pi_1(\mathsf{M}) \cong \pi_2(X)$, $\pi_0(\mathsf{M}) \cong \pi_1(X)$.

Mac Lane and Whitehead, [95], showed that crossed modules give algebraic models for all homotopy 2-types of connected spaces. We will visit this result in more detail later, but loosely a 2-equivalence between spaces is a continuous map that induces isomorphisms on π_1 and π_2, the first two homotopy groups. Two spaces have the same 2-type if there is a zig-zag of 2-equivalences joining them.

2.1.3 Restriction along a homomorphism φ/ 'Change of base'

Given a crossed module, (C, H, ∂), over H and a homomorphism $\varphi : G \to H$, we can form the pullback:

$$\begin{tikzcd} D \arrow[r, "\psi"] \arrow[d, "\partial'"'] & C \arrow[d, "\partial"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

in $Grps$. Clearly the universal property of pullbacks gives a good universal property for this, namely that any morphism $(\varphi', \varphi) : (C', G, \delta) \to (C, H, \partial)$ factors uniquely through (ψ, φ) and a morphism in $CMod_G$ from (C', G, δ) to (D, G, ∂'). Of course this statement depends on verification that (D, G, ∂') is a crossed module and that the resulting maps are morphisms of crossed modules, but this is routine, and will be **left as an exercise**. (You may need to recall that D can be realised, up to isomorphism, as $G \times_H C = \{(g, c) \mid \varphi(g) = \partial c\}$. It is for you to see what the action is.)

This construction also behaves nicely on morphisms of crossed modules over H and yields a functor,

$$\varphi^* : CMod_H \to CMod_G,$$

which will be called *restriction along* φ.

We next turn to the use of crossed modules in combinatorial group theory.

2.2 Group presentations, identities and 2-syzygies

2.2.1 Presentations and Identities

(cf. Brown-Huebschmann, [32]) We consider a presentation, $\mathcal{P} = (X : R)$, of a group G. The elements of X are called *generators* and those of R *relators*. We then have a short exact sequence,

$$1 \to N \to F \to G \to 1,$$

where $F = F(X)$, the free group on the set X, R is a subset of F and $N = N(R)$ is the normal closure in F of the set R.

A standard if somewhat trivial example is given by the *standard presentation* of a group, G. We take $X = \{x_g \mid g \in G, g \neq 1\}$, to be a set in bijective correspondence with the underlying set of G. (You can take X equal to that set if you like, but sometimes it is better to have a distinct set, for instance, it make for an easier notation for the description of certain morphisms.) The set of relations will be $R = \{x_g.x_h = x_{gh} \mid g, h \in G\}$.

The group F acts on N by conjugation: ${}^u c = ucu^{-1},\ c \in N, u \in F$ and the elements of N are words in the conjugates of the elements of R:

$$c = {}^{u_1}(r_1^{\varepsilon_1})^{u_2}(r_2^{\varepsilon_2})\dots {}^{u_n}(r_n^{\varepsilon_n})$$

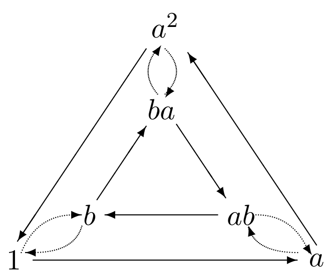
where each ε_i is $+1$ or -1. One also says such elements are *consequences* of R. Heuristically an *identity among the relations* of $\mathcal{P}$ is such an element c which equals 1. The problem of what this means is analogous to that of working with a relation in R. For example, in the presentation $(a : a^3)$ of C_3, the cyclic group of order 3, if a is thought of as being an element of C_3, then $a^3 = 1$, so why is this different from the situation with the 'presentation', $(a : a = 1)$? To get around that difficulty the free group on the generators $F(X)$ was introduced and, of course, in $F(\{a\})$, a^3 is not 1. A similar device, namely *free crossed modules* on the presentation will be introduced in a moment to handle the identities. Before that consider some examples which indicate that identities exist even in some quite common-or-garden cases.

Example 1: Suppose $r \in R$, but it is a power of some element $s \in F$, i.e. $r = s^m$. Of course, $rs = sr$ and

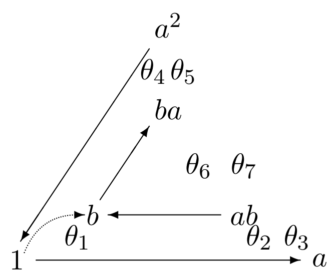
$${}^s r r^{-1} = 1$$

so ${}^s r.r^{-1}$ is an identity. In fact, there will be a unique $z \in F$ with $r = z^q$, q maximal with this property. This z is called the *root of* r and if $q > 1$, r is called a *proper power*.

Example 2: Consider one of the standard presentations of S_3, $(a, b : a^3, b^2, (ab)^2)$. Write $r = a^3$, $s = b^2$, $t = (ab)^2$. Here the presentation leads to F, free of rank 2, but $N(R) \subset F$, so it must be free as well, by the Nielsen-Schreier theorem. Its rank will be 7, given by the Schreier index formula or, geometrically, it will be the fundamental group of the Cayley graph of the presentation. This group is free on generators corresponding to edges outside a maximal tree as in the following diagram:



The Cayley graph of S_3



and a maximal tree in it.

The set of normal generators of $N(R)$ has 3 elements; $N(R)$ is free on 7 elements (corresponding to the edges not in the tree), but is specified as consisting of products of conjugates of r, s and t, and there are infinitely many of these. Clearly there must be some slight redundancy, i.e., there must be some identities among the relations!

A path around the outer triangle corresponds to the relation r; each other region corresponds to a conjugate of one of r, s or t. (It may help in what follows to think of the graph being embedded on a 2-sphere, so 'outer' and 'outside' mean 'round the back face.) Consider a loop around a region. Pick a path to a start vertex of the loop, starting at 1. For instance the path that leaves 1 and goes along a, b and then goes around aaa before returning by $b^{-1}a^{-1}$ gives $abrb^{-1}a^{-1}$. Now the path around the outside can be written as a product of paths around the inner parts of the graph, e.g. $(abab)b^{-1}a^{-1}b^{-1}(bb)(b^{-1}a^{-1}b^{-1}a^{-1})\ldots$ and so on. Thus r can be written in a non-trivial way as a product of conjugates of r, s and t. (An explicit identity constructed like this is given in [32].)

Example 3: In a presentation of the free Abelian group on 3 generators, one would expect the commutators, $[x, y]$, $[x, z]$ and $[y, z]$. The well-known identity, usually called the Jacobi identity, expands out to give an identity among these relations (again see [32], p.154 or Loday, [90].)

2.2.2 Free crossed modules and identities

The idea that an identity is an equation in conjugates of relations leads one to consider formal conjugates of symbols that label relations. Abstracting this a bit, suppose G is a group and $f : Y \to G$, a function 'labelling' the elements of some subset of G. To form a conjugate, you need a thing being conjugated and an element 'doing' the conjugating, so form pairs (p, y), $p \in G$, $y \in Y$, to be thought of as ${}^py$, the *formal conjugate* of y by p. Consequences are words in conjugates of relations, *formal consequences* are elements of $F(G \times Y)$. There is a function extending f from $G \times Y$ to G given by

$$\bar{f}(p, y) = pf(y)p^{-1},$$

converting a formal conjugate to an actual one and this extends further to a group homomorphism

$$\varphi : F(G \times Y) \to G$$

defined to be $\bar{f}$ on the generators. The group G acts on the left on $G \times Y$ by multiplication: $p.(p', y) = (pp', y)$. This extends to a group action of G on $F(G \times Y)$. For this action, φ is G-equivariant if G is given its usual G-group structure by conjugations / inner automorphisms. Naively identities are the elements in the kernel of this, but there are some elements in that kernel that are there regardless of the form of function f. In particular, suppose that $g_1, g_2 \in G$ and $y_1, y_2 \in Y$ and look at

$$(g_1, y_1)(g_2, y_2)(g_1, y_1)^{-1}((g_1f(y_1)g_1^{-1})g_2, y_2)^{-1}.$$

Such an element is always annihilated by φ. The normal subgroup generated by such elements is called the Peiffer subgroup. We divide out by it to obtain a quotient group. This is the construction of the free crossed module on the function f. If f is, as in our initial motivation, the inclusion of a set of relators into the free group on the generators we call the result the *free crossed module on the presentation* $\mathcal{P}$ and denote it by $C(\mathcal{P})$.

We can now formally define the module of identities of a presentation $\mathcal{P} = (X : R)$. We form the free crossed module on $R \to F(X)$, which we will denote by $\partial : C(\mathcal{P}) \to F(X)$. The *module* *of identities* of $\mathcal{P}$ is $Ker\,\partial$. By construction, the group presented by $\mathcal{P}$ is $G \cong F(X)/Im\,\partial$, where $Im\,\partial$ is just the normal closure of the set, R, of relations and we know that $Ker\,\partial$ is a G-module. We will usually denote the module of identities by $\pi_{\mathcal{P}}$.

We can get to $C(\mathcal{P})$ in another way. Construct a space from the combinatorial information in $C(\mathcal{P})$ as follows. Take a bunch of circles labelled by the elements of X; call it $K(\mathcal{P})_1$, it is the 1-skeleton of the space we want. We have $\pi_1(K(\mathcal{P})_1 \cong F(X)$. Each relator $r \in R$ is a word in X so gives us a loop in $K(\mathcal{P})_1$, following around the circles labelled by the various generators making up r. This loop gives a map $S^1 \xrightarrow{f_r} K(\mathcal{P})_1$. For each such r we use f_r to glue a 2-dimensional disc e^2_r to $K(\mathcal{P})_1$ yielding the space $K(\mathcal{P})$. The crossed module $C(\mathcal{P})$ is isomorphic to $\pi_2(K(\mathcal{P}), K(\mathcal{P})_1) \xrightarrow{\partial} \pi_1(K(\mathcal{P})_1$.

The main problem is how to calculate $\pi_{\mathcal{P}}$ or equivalently $\pi_2(K(\mathcal{P}))$. One approach is via an associated chain complex. This can be viewed as the chains on the universal cover of $K(\mathcal{P})$, but can also be defined purely algebraically, for which see Brown-Huebschmann, [32], or Loday, [90]. That algebraic - homological approach leads to 'homological syzygies'. For the moment we will concentrate on:

2.3 Cohomology, crossed extensions and algebraic 2-types

2.3.1 Cohomology and extensions, continued

Suppose we have any group extension

$$\mathcal{E}: \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

with K Abelian, but not necessarily central. We can look at various possibilities.

If we can split p, by a homomorphism $s : G \to E$, with $ps = Id_G$, then, of course, $E \cong K \rtimes G$ by the isomorphisms,

$$e \longrightarrow (esp(e)^{-1}, p(e)),$$
$$ks(g) \longleftarrow (k, g),$$

which are compatible with the projections etc., so there is an equivalence of extensions

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\cong"] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & K \rtimes G \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

Our convention for multiplication in $K \rtimes G$ will be

$$(k, g)(k', g') = (k^g k', gg').$$

But what if p does not split. We can build a (small) category of extensions $\mathcal{E}xt(G, K)$ with objects such as $\mathcal{E}$ above and in which a morphism from $\mathcal{E}$ to $\mathcal{E}'$ is a diagram

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & E' \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

By the 5-lemma, α will be an isomorphism, so $\mathcal{E}xt(G,K)$ is a groupoid.

In $\mathcal{E}$, the epimorphism p is usually not splittable, but as a function between sets, it is onto so we can pick an element in each $p^{-1}(g)$ to get a transversal (or set of coset representatives), $s : G \to E$. We get a comparison pairing / obstruction map or 'factor set' :

$$f : G \times G \to E$$

$$f(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1},$$

which will be trivial, (i.e., $f(g_1, g_2) = 1$ for all $g_1, g_2 \in G$) exactly if s splits p, i.e., if s is a homomorphism. This construction assumes that we know the multiplication in E, otherwise we cannot form this product! On the other hand given this 'f', we can work out the multiplication. As a set, E will be the product $K \times G$, identified with it by the same formulae as in the split case, noting that $pf(g_1, g_2) = 1$, so 'really' we should think of f as ending up in the subgroup K, then we have

$$(k_1, g_1)(k_2, g_2) = ({k_1}^{s(g_1)}k_2f(g_1, g_2), g_1g_2).$$

The product is *twisted* by the pairing f. Of course, we need this multiplication to be associative and, to ensure that, f must satisfy a cocycle condition:

$${}^{s(g_1)}f(g_2, g_3)f(g_1, g_2g_3) = f(g_1, g_2)f(g_1g_2, g_3).$$

This is a well known formula from group cohomology, more so if written additively:

$$s(g_1)f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2) = 0.$$

Here we actually have various parts of the nerve of G involved in the formula. The group G 'is' a small category (groupoid with one object), which we will, for the moment, denote $\mathcal{G}$. The triple $\sigma = (g_1, g_2, g_3)$ is a 3-simplex in $Ner(\mathcal{G})$ and its faces are

$$\begin{aligned} d_0\sigma &= (g_2, g_3), \\ d_1\sigma &= (g_1g_2, g_3), \\ d_2\sigma &= (g_1, g_2g_3), \\ d_3\sigma &= (g_1, g_2). \end{aligned}$$

This is all very classical. We can use it in the usual way to link $\pi_0(\mathcal{E}xt(G,K))$ with $H^2(G,K)$ and so is the 'modern' version of Schreier's theory of group extensions, at least in the case that K is Abelian.

For a long time there was no obvious way to look at the elements of $H^3(G,K)$ in a similar way. In Mac Lane's homology book, [92], you can find a discussion from the classical viewpoint. In Brown's [24], the link with crossed modules is sketched although no references for the details are given, for which see Mac Lane's [94].

If we have a crossed module $C \xrightarrow{\partial} P$, then we saw that $Ker\,\partial$ is central in C and is a $P/\partial C$-module. We thus have a 'crossed 2-fold extension':

$$K \xrightarrow{i} C \xrightarrow{\partial} P \xrightarrow{p} G,$$

where $K = Ker\,\partial$ and $G = P/\partial C$. (We will write $N = \partial C$.)

Repeat the same process as before for the extension

$$N \to P \to G,$$

but take extra care as N is usually not Abelian. Pick a transversal $s : G \to P$ giving $f : G\times G \to N$ as before (even with the same formula). Next look at

$$K \xrightarrow{i} C \to N,$$

and lift f to C via a choice of $F(g_1, g_2) \in C$ with image $f(g_1, g_2)$ in N.

The pairing f satisfied the cocycle condition, but we have no means of ensuring that F will do so, i.e. there will be, for each triple (g_1, g_2, g_3), an element $c(g_1, g_2, g_3) \in C$ such that

$${}^{s(g_1)}F(g_2, g_3)F(g_1, g_2g_3) = i(c(g_1, g_2, g_3))F(g_1, g_2)F(g_1g_2, g_3),$$

and some of these $c(g_1, g_2, g_3)$ may be non-trivial. The $c(g_1, g_2, g_3)$ will satisfy a cocycle condition correspond to a 4-simplex in $Ner(\mathcal{G})$, and one can reconstruct the crossed 2-fold extension up to equivalence from F and c. Here 'equivalence' is generated by maps of 'crossed' exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & C \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & K \arrow[r] & C' \arrow[r] & P' \arrow[r] & G \arrow[r] & 1,
\end{tikzcd}$$

but these morphisms need not be isomorphisms. Of course, this identifies $H^3(G, K)$ with π_0 of the resulting category.

What about $H^4(G, K)$? Yes, something similar works, but we do not have the machinery to do it here, yet.

2.3.2 Not really an aside!

Suppose we start with a crossed module $\mathsf{C} = (C, P, \partial)$. We can build an internal category, $\mathcal{X}(\mathsf{C})$, in *Grps* from it. The group of objects of $\mathcal{X}(\mathsf{C})$ will be P and the group of arrows $C \rtimes P$. The source map

$$s : C \rtimes P \to P \quad \text{is} \quad s(c, p) = p,$$

the target

$$t : C \rtimes P \to P \quad \text{is} \quad t(c, p) = \partial c.p.$$

(That looks a bit strange. That sort of construction usually does not work, multiplying two homomorphisms together is a recipe for trouble! - but it does work here:

$$\begin{aligned} t((c_1, p_1).(c_2, p_2)) &= t(c_1{}^{p_1}c_2, p_1p_2) \\ &= \partial(c_1{}^{p_1}c_2).p_1p_2, \end{aligned}$$

whilst $t(c_1, p_1).t(c_2, p_2) = \partial c_1.p_1.\partial c_2.p_2$, but remember $\partial(c_1{}^{p_1}c_2) = \partial c_1.p_1.\partial c_2.p_1^{-1}$, so they are equal.)

The identity morphism is $i(p) = (1, p)$, but what about the composition. Here it helps to draw a diagram. Suppose $(c_1, p_1) \in C \rtimes P$, then it is an arrow

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1,$$

and we can only compose it with (c_2, p_2) if $p_2 = \partial c_1.p_1$. This gives

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1 \xrightarrow{(c_2,\partial c_1.p_1)} \partial c_2 \partial c_1.p_1.$$

The obvious candidate for the composite arrow is (c_2c_1, p_1) and it works!

In fact, $\mathcal{X}(\mathsf{C})$ is an internal groupoid as $(c_1^{-1}, \partial c_1.p_1)$ is an inverse for (c_1, p_1).

Now if we started with an internal category

$$\begin{tikzcd} G_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & G_0 \arrow[l, shift left=2, "i"] \end{tikzcd} ,$$

etc., then set $P = G_0$ and $C = Ker\, s$ with $\partial = t\,|_C$ to get a crossed module.

Theorem 2 *(Brown-Spencer,[34]) The category of crossed modules is equivalent to that of internal categories in Grps.* ■

You have, almost, seen the proof. As beginning students of algebra, you learnt that equivalence relations on groups need to be congruence relations for quotients to work well and that congruence relations 'are the same as' normal subgroups. That is the essence of the proof needed here, but we have groupoids rather than equivalence relations and crossed modules rather than normal subgroups.

Of course, any morphism of crossed modules has to induce an internal functor between the corresponding internal categories and *vice versa*. That is a **good exercise** for you to check that you have understood the link that the Brown-Spencer theorem gives.

This is a good place to mention 2-groups. The notion of 2-category is one that should be fairly clear even if you have not met it before. For instance, the category of small categories, functors and natural transformations is a 2-category. Between each pair of objects, we have not just a set of functors as morphisms but a small category of them with the natural transformations between them as the arrows in this second level of structure. The notion of 2-category is abstracted from this. We will not give a formal definition here (but suggest that you look one up if you have not met the idea before). A 2-category thus has objects, arrows or morphisms (or sometimes '1-cells') between them and then some 2-cells (sometimes called '2-arrows' or '2-morphisms') between them.

Definition: A *2-groupoid* is a 2-category in which all 1-cells and 2-cells are invertible.

If the 2-groupoid has just one object then we call it a *2-group*.

Of course, there are also 2-functors between 2-categories and so, in particular, between 2-groups. Again this is for **you to formulate, looking up relevant definitions, etc.**

Internal categories in *Grps* are really exactly the same as 2-groups. The Brown-Spencer theorem thus constructs the *associated 2-group of a crossed module*. The fact that the composition in the internal category must be a group homomorphism implies that the '*interchange law*' must hold. This equation is in fact equivalent via the Brown-Spencer result to the Peiffer identity. (It is **left to you** to find out about the interchange law and to check that it is the Peiffer axiom in disguise. We will see it many times later on.)

Here would be a good place to mention that an internal monoid in *Grps* is just an Abelian group. The argument is well known and is usually known by the name of the *Eckmann-Hilton argument*.

This starts by looking at the interchange law, which states that the monoid multiplication must be group homomorphism. From this it derives that the monoid identity must also be the group identity and that the two compositions must coincide. It is then easy to show that the group is Abelian.

2.3.3 Perhaps a bit more of an aside ... for the moment!

This is quite a good place to mention the groupoid based theory of all this. The resulting objects look like abstract 2-categories and are 2-groupoids. We have a set of objects, K_0, a set of arrows, K_1, depicted $x \xrightarrow{p} y$, and a set of two cells

$$\begin{tikzcd} x \arrow[r, bend left, "p"] \arrow[r, bend right, "\partial c.p"'] \arrow[r, phantom, "\Downarrow(c,p)"] & y \end{tikzcd} .$$

In our previous diagrams, as all the elements of P started and ended at the same single object, we could shift dimension down one step; our old objects are now arrows and our old arrows are 2-cells. We will return to this later.

The important idea to note here is that a 'higher dimensional category' has a link with an algebraic object. The 2-group(oid) provides a useful way of interpreting the structure of the crossed module *and* indicates possible ways towards similar applications and interpretations elsewhere. For instance, a presentation of a monoid leads more naturally to a 2-category than to any analogue of a crossed module, since kernels are less easy to handle than congruences in Mon.

There are other important interpretations of this. Categories such as that of vector spaces, Abelian groups or modules over a ring, have an additional structure coming from the tensor product, $A \otimes B$. They are *monoidal categories*. One can 'multiply' objects together and this is linked to a related multiplication on morphisms between the objects. In many of the important examples the multiplication is not strictly associative, so for instant, if A, B, C are objects there is an isomorphism between $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$, but this isomorphism is most definitely not the identity as the two objects are constructed in different ways. A similar effect happens in the category of sets with ordinary Cartesian product. The isomorphism is there because of universal properties, but it is again not the identity. It satisfies some coherence conditions, (a cocycle condition in disguise), relating to associativity of four fold tensors and the associahedron that we gave earlier, is a corresponding diagram for the five fold tensors. (Yes, there is a strong link, but that is not for these notes!) Our 2-group(oid) is the 'suspension' or 'categorification' of a similar structure. We can multiply objects and 'arrows' and the result is a strict 'gr-groupoid', or 'categorical group', i.e. a strict monoidal category with inverses. This is vague here, but will gradually be explored later on. If you want to explore the ideas further now, look at Baez and Dolan, [7].

(At this point, you do not need to know the definition of a monoidal category, but **remember to look it up in the not too distance future**, if you have not met it before, as later on the insights that an understanding of that notion gives you, will be very useful. It can be found in many places in the literature, and on the internet. The approach that you will get on best with depends on your background and your likes and dislikes mathematically, so we will not give one here.)

Just as associativity in a monoid is replaced by a 'lax' associativity 'up to coherent isomorphisms' in the above, gr-groupoids are 'lax' forms of internal categories in groups and thus indicate the presence of a crossed module-like structure, albeit in a weakened or 'laxified' form. Later we will see naturally occurring gr-groupoid structures associated with some constructions in non-Abelian cohomology. There is also a sense in which the link between fibrations and crossed modules given earlier here, indicates that fibrations are like a related form of lax crossed modules. In the notion of fibred category and the related Grothendieck construction, this intuition begins to be 'solidified' into a clearer strong relationship.

2.3.4 Automorphisms of a group yield a 2-group

We could also give this section a subtitle:

The automorphisms of a 1-type give a 2-type.

This is really an extended exercise in playing around with the ideas from the previous two sections. It uses a small amount of categorical language, but, hopefully, in a way that should be easy for even a categorical debutant to follow. The treatment will be quite detailed as it is that detail that provides the links between the abstract and the concrete.

We start with a look at 'functor categories', but with groupoids rather than general small categories as input. Suppose that $\mathcal{G}$ and $\mathcal{H}$ are groupoids, then we can form a new groupoid, $\mathcal{H}^{\mathcal{G}}$, whose objects are the functors, $f : \mathcal{G} \to \mathcal{H}$. Of course, functors in this context are just morphisms of groupoids, and, if $\mathcal{G}$, and $\mathcal{H}$ are $G[1]$ and $H[1]$, that is, two groups, G and H, thought of as one object groupoids, then the objects of $\mathcal{H}^{\mathcal{G}}$ are just the homomorphisms from G to H thought of in a slightly different way.

That gives the objects of $\mathcal{H}^{\mathcal{G}}$. For the morphisms from f_0 to f_1, we 'obviously' should think of natural transformations. (As usual, if you are not sufficiently conversant with elementary categorical ideas, pause and look them up in a suitable text of in Wikipedia.) Suppose $\eta : f_0 \to f_1$ is a natural transformation, then, for each x, an object of $\mathcal{G}$, we have an arrow,

$$\eta(x) : f_0(x) \to f_1(x),$$

in $\mathcal{H}$ such that, if $g : x \to y$ in $\mathcal{G}$, then the square

$$\begin{tikzcd}
f_0(x) \arrow[r, "\eta(x)"] \arrow[d, "f_0(g)"'] & f_1(x) \arrow[d, "f_1(g)"] \\
f_0(y) \arrow[r, "\eta(y)"'] & f_1(y)
\end{tikzcd}$$

commutes, so η 'is' the family, $\{\eta(x) \mid x \in Ob(\mathcal{G}\}$. Now assume $\mathcal{G} = G[1]$ and $\mathcal{H} = H[1]$, and that we try to interpret $\eta(x) : f_0(x) \to f_1(x)$ back down at the level of the groups, that is, a bit more 'classically' and group theoretically. There is only one object, which we denote $*$, if we need it, so we have that η corresponds to a single element, $\eta(*)$, in H, which we will write as h for simplicity, but now the condition for commutation of the square just says that, for any element $g \in G$,

$$hf_0(g) = f_1(g)h,$$

i.e., that f_0 and f_1 are *conjugate* homomorphisms, $f_1 = hf_0h^{-1}$..

It should be clear, (but **check that it is**), that this definition of morphism makes $\mathcal{H}^{\mathcal{G}}$ into a category, in fact into a groupoid, as the morphisms compose correctly and have inverses. (To get the inverse of η take the family $\{\eta(x)^{-1} \mid x \in Ob(\mathcal{G}\}$ and check the relevant squares commute.)

So far we have 'proved':

Lemma 6 *For groupoids, $\mathcal{G}$ and $\mathcal{H}$, the functor category, $\mathcal{H}^{\mathcal{G}}$, is a groupoid.* ■

We will be a bit sloppy in notation and will write H^G for what should, more precisely, be written $H[1]^{G[1]}$.

We note that it is usual to observe that, for Abelian groups, A, and B, the set of homomorphisms from A to B is itself an Abelian group, but that the set of homomorphisms from one non-Abelian group to another has no such nice structure. Although this is sort of true, the point of the above is that that set forms the set of objects for a very neat algebraic object, namely a groupoid!

If we have a third groupoid, $\mathcal{K}$, then we can also form $\mathcal{K}^{\mathcal{H}}$ and $\mathcal{K}^{\mathcal{G}}$, etc. and, as the objects of $\mathcal{K}^{\mathcal{H}}$ are homomorphisms from $\mathcal{H}$ to $\mathcal{K}$, we might expect to compose with the objects of $\mathcal{H}^{\mathcal{G}}$ to get ones of $\mathcal{K}^{\mathcal{G}}$. We might thus hope for a composition *functor*

$$\mathcal{K}^{\mathcal{H}} \times \mathcal{H}^{\mathcal{G}} \to \mathcal{K}^{\mathcal{G}}.$$

(There are **various things to check**, but we need not worry. We are really working with functors and natural transformations and with the investigation that shows that the category of small categories is 2-category. This means that if you get bogged down in the detail, you can easily find the ideas discussed in many texts on category theory.) This works, so we have that the category, *Grpds* has also a 2-category structure. (It is a '*Grpds*-enriched' category; see later for enriched categories. The formal definition is in section **??**, although the basic idea is used before that.)

We need to recall next that in any category, $\mathcal{C}$, the endomorphisms of any object, X, form a monoid, $End(X) := \mathcal{C}(X, X)$. You just use the composition and identities of $\mathcal{C}$ 'restricted to X'. If we play that game with any groupoid enriched category, C, then for any object, X, we will have a groupoid, $\mathsf{C}(X, X)$, which we might write $\mathsf{End}(X)$, (that is, using the same font to indicate 'enriched') and which also has a monoid structure,

$$\mathsf{C}(X, X) \times \mathsf{C}(X, X) \to \mathsf{C}(X, X).$$

It will be a monoid internal to *Grpds*. In particular, for any groupoid, $\mathcal{G}$, we have such an internal monoid of endomorphisms, $\mathcal{G}^{\mathcal{G}}$, and specialising down even further, for any group, G, such an internal monoid, G^G. Note that this is internal to the category of *groupoids* not of groups, as its monoid of objects is the endomorphism monoid of G, not a single element set. Within G^G, we can restrict attention to the subgroupoid on the automorphisms of G. We thus have this groupoid, $\mathsf{Aut}(G)$, which has as objects the automorphisms of G and, as typical morphism, $\eta : f_0 \to f_1$, a conjugation. It is important to note that as η is specified by an element of G and an automorphism, f_0, of G, the pair, (g, f_0), may then be a good way of thinking of it. (Two points, that may be obvious, but are important even if they are, are that the morphism η is not conjugation itself, but conjugates f_0. One has to specify where this morphism starts, its domain, as well as what it does, namely conjugate by g. Secondly, in (g, f_0), we *do* have the information on the codomain of η, as well. It is $gf_0g^{-1} = f_1$.)

Using this basic notation for the morphisms, we will look at the various bits of structure this thing has. (Remember, $\eta : f_0 \to f_1$ and $f_1 = gf_0g^{-1}$, as we will need to use that several times.) We have compositions of these pairs in two ways:

(a) as natural transformations: if

$$\begin{array}{rll} & \eta : f_0 \to f_1, & \eta = (g, f_0), \\ \text{and} & \eta' : f_1 \to f_2, & \eta' = (g', f_1), \end{array}$$

then the composite is $\eta' \sharp_1 \eta = (g'g, f_0)$. (That is easy to check. As, for instance, $f_2 = g' f_1 (g')^{-1} = (g'g) f_0 (g'g)^{-1}, \ldots,$ it all works beautifully). (A word of **warning** here, $(g'g)f_0(g'g)^{-1}$ is the conjugate of the automorphism f_0 by the element $(g'g)$. The bracket does not refer to f_0 applied to the 'thing in the bracket', so, for $x \in G$, $((g'g)f_0(g'g)^{-1})(x)$ is, in fact, $(g'g)f_0(x)(g'g)^{-1}$. This *is* slightly confusing so think about it, so as not to waste time later in avoidable confusion.)

b) using composition, $\sharp_0$, in the monoid structure. To understand this, it is easier to look at that composition as being specialised from the one we singled out earlier,

$$\mathcal{K}^{\mathcal{H}} \times \mathcal{H}^{\mathcal{G}} \to \mathcal{K}^{\mathcal{G}},$$

which is the composition in the 2-category of groupoids. (We really want $\mathcal{G} = \mathcal{H} = \mathcal{K}$, but, by keeping the more general notation, it becomes easier to see the roles of each $\mathcal{G}$.)

We suppose $f_0, f_1 : \mathcal{G} \to \mathcal{H}$, $f'_0, f'_1 : \mathcal{G} \to \mathcal{H}$, and then $\eta : f_0 \to f_1$, $\eta' : f'_0 \to f'_1$. The 2-categorical picture is

$$\begin{tikzcd} \cdot \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Downarrow\eta"] & \cdot \arrow[r, bend left, "f'_0"] \arrow[r, bend right, "f'_1"'] \arrow[r, phantom, "\Downarrow\eta'"] & \cdot \end{tikzcd} \quad = \quad \begin{tikzcd} \cdot \arrow[r, bend left, "f'_0 f_0"] \arrow[r, bend right, "f'_1 f_1"'] \arrow[r, phantom, "\Downarrow\eta''"] & \cdot \end{tikzcd},$$

with η'' being the desired composite, $\eta' \sharp_0 \eta$, but how is it calculated. The important point is the *interchange law*. We can 'whisker' on the left or right, or, since the 'left-right' terminology can get confusing (does 'left' mean 'diagrammatically' or 'algebraically' on the left?), we will often use 'pre-' and 'post-' as alternative prefixes. The terminology may seem slightly strange, but is quite graphic when suitable diagrams are looked at! Whiskering corresponds to an interaction between 1-cell and 2-cells in a 2-category. In 'post-whiskering', the result is the composite of a 2-cell *followed* by a 1-cell:

Post-whiskering:

$$f'_0 \sharp_0 \eta : f'_0 \sharp_0 f_0 \to f'_0 \sharp_0 f_1,$$

$$\begin{tikzcd} \cdot \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Downarrow\eta"] & \cdot \arrow[r, "f'_0"] & \cdot \end{tikzcd}$$

(It is convenient, here, to write the more formal $f'_0 \sharp_0 f_0$, for what we would usually write as $f'_0 f_0$.) The natural transformation, η is given by a family of arrows in $\mathcal{H}$, so $f'_0 \sharp_0 \eta$ is given by mapping that family across to $\mathcal{K}$ using f'_0. (Specialising to $\mathcal{G} = \mathcal{H} = \mathcal{K} = G[1]$, if $\eta = (g, f_0)$, then $f'_0\sharp_0\eta = (f'_0(g), f'_0 f_0)$, as is easily checked; similarly for $f'_1\sharp_0\eta$.)

Pre-whiskering:

$$\eta'\sharp_0 f_0 : f'_0\sharp_0 f_0 \to f'_1\sharp_0 f_0,$$

$$\begin{tikzcd} \cdot \arrow[r, "f_0"] & \cdot \arrow[r, bend left, "f'_0"] \arrow[r, bend right, "f'_1"'] \arrow[r, phantom, "\Downarrow\eta'"] & \cdot \end{tikzcd} .$$

Here the morphism f_0 does not influence the g-part of η' at all. It just alters the domains. In the case that interests us, if $\eta' = (g', f'_0)$, then $\eta'\sharp_0 f_0 = (g', f'_0 f_0)$.

The way of working out $\eta'\sharp_0\eta$ is by using $\sharp_1$-composites. First,

$$\eta'\sharp_0\eta : f'_0 f_0 \to f'_1 f_1,$$

and we can go

$$\eta'\sharp_0 f_0 : f'_0 f_0 \to f'_1 f_0,$$

and then, to get to where we want to be, that is, $f'_1 f_1$, we use

$$f'_1\sharp_0\eta : f'_1 f_0 \to f'_1 f_1.$$

This uses the $\sharp_1$-composition, so

$$\begin{aligned} \eta'\sharp_0\eta &= (f'_1\sharp_0\eta)\sharp_1(\eta'\sharp_0 f_0) \\ &= (f'_1(g), f'_1 f_0)\sharp_1(g', f'_0 f_0) \\ &= (f'_1(g).g', f'_0 f_0), \end{aligned}$$

but $f'_1(g) = g' f_0(g)(g')^{-1}$, so the end results simplifies to $(g' f_0(g), f'_0 f_0)$. **Hold on!** That looks nice, but we could have also calculated $\eta'\sharp_0\eta$ using the other form as the composite,

$$\begin{aligned} \eta'\sharp_0\eta &= (\eta'\sharp_0 f_1)\sharp_1(f'_0\sharp_0\eta) \\ &= (g', f'_0 f_1)\sharp_1(f'_0(g), f'_0 f_0) \\ &= (g' f'_0(g), f'_0 f_0), \end{aligned}$$

so we did not have any problem. (All the properties of an internal groupoid in *Grps*, or, if you prefer that terminology, 2-group, can be derived from these two compositions. The $\sharp_1$ composition is the 'groupoid' direction, whilst the $\sharp_0$ is the 'group' one.)

We thus have a group of natural transformations made up of pairs, (g, f_0) and whose multiplication is given as above. This is just the semi-direct product group, $G \rtimes Aut(G)$, for the natural and obvious action of $Aut(G)$ on G. This group is sometimes called the *holomorph* of G.

We have two homomorphisms from $G \rtimes Aut(G)$ to $Aut(G)$. One sends (g, f_0) to f_0, so is just the projection, the other sends it to $f_1 = g f_0 g^{-1} = \iota_g \circ f_0$. We can recognise this structure as being the associated 2-group of the crossed module, $(G, Aut(G), \iota)$, as we met on page 36. We call $\mathsf{Aut}(G)$, the *automorphism 2-group* of G..

2.3.5 Back to 2-types

From our crossed module, $\mathsf{C} = (C, P, \partial)$, we can build the internal groupoid, $\mathcal{X}(\mathsf{C})$, as before, then apply the nerve construction internally to the internal groupoid structure to get a simplicial group, $K(\mathsf{C})$.

Definition: Given a crossed module, $\mathsf{C} = (C, P, \partial)$, the nerve (taken internally in *Grps*) of the internal groupoid, $\mathcal{X}(\mathsf{C})$, defined by C, will be called the nerve of C or, if more precision is needed, its *simplicial group nerve* and will be denoted $K(\mathsf{C})$.

The simplicial set, $\overline{W}(K(\mathsf{C}))$, or its geometric realisation, would be called the *classifying space* of C.

We need this in some detail in low dimensions.

$$\begin{array}{rcll} K(\mathsf{C})_0 & = & P & \\ K(\mathsf{C})_1 & = & C \rtimes P & \qquad\qquad d_0 = t, d_1 = s \\ K(\mathsf{C})_2 & = & C \rtimes (C \rtimes P), & \end{array}$$

where $d_0(c_2, c_1, p) = (c_2, \partial c_1.p)$, $d_1(c_2, c_1, p) = (c_2.c_1, p)$ and $d_2(c_2, c_1, p) = (c_1, p)$. The pattern continues with $K(\mathsf{C})_n = C \rtimes (\ldots \rtimes (C \rtimes P)\ldots)$, having n-copies of C. The d_i, for $0 < i < n$, are given by multiplication in C, d_0 is induced from t and d_n is a projection. The s_i are insertions of identities. (We will examine this in more detail later.)

Remark: A word of caution: for G a group considered as a crossed module, this 'nerve' is not the nerve of G in the sense used earlier. It is just the constant simplicial group corresponding to G. What is often called the nerve of G is what here has been called its classifying space. One way to view this is to note that $\mathcal{X}(\mathsf{C})$ has two independent structures, one a group, the other a category, and *this* nerve is of the category structure. The group, G, considered as a crossed module is like a set considered as a (discrete) category, having only identity arrows.)

The Moore complex of $K(\mathsf{C})$ is easy to calculate and is just $NK(\mathsf{C})_i = 1$ if $i \geq 2$; $NK(\mathsf{C})_1 \cong C$; $NK(\mathsf{C})_0 \cong P$ with the $\partial : NK(\mathsf{C})_1 \to NK(\mathsf{C})_0$ being exactly the given ∂ of C. (This is left as an exercise. It is a useful one to do in detail.)

Proposition 4 *(Loday, [89]) The category CMod of crossed modules is equivalent to the subcategory of Simp.Grps, consisting of those simplicial groups, G, having Moore complexes of length 1, i.e.* $NG_i = 1$ *if* $i \geq 2$. ■

This raises the interesting question as to whether it is possible to find alternative algebraic descriptions of the structures corresponding to Moore complexes of length n.

Is there any way of going directly from simplicial groups to crossed modules? Yes. The last two terms of the Moore complex will give us:

$$\partial : NG_1 \to NG_0 = G_0$$

and G_0 acts on NG_1 by conjugation via s_0, i.e. if $g \in G_0$ and $x \in NG_1$, then $s_0(g)xs_0(g)^{-1}$ is also in NG_1. (Of course, we could use multiple degeneracies to make g act on an $x \in NG_n$ just as easily.) As $\partial = d_0$, it respects the G_0 action, so CM1 is satisfied. In general, CM2 will not be satisfied. Suppose $g_1, g_2 \in NG_1$ and examine ${}^{\partial g_1}g_2 = s_0d_0g_1.g_2.s_0d_0g_1^{-1}$. This is rarely equal to $g_1g_2g_1^{-1}$. We write $\langle g_1, g_2\rangle = [g_1, g_2][g_2, s_0d_0g_1] = g_1g_2g_1^{-1}.({}^{\partial g_1}g_2)^{-1}$, so it measures the obstruction to CM2 for this pair g_1, g_2. This is often called the *Peiffer commutator* of g_1 and g_2. Noting that $s_0d_0 = d_0s_1$, we have an element

$$\{g_1, g_2\} = [s_0g_1, s_0g_2][s_0g_2, s_1g_1] \in NG_2$$

and $\partial\{g_1, g_2\} = \langle g_1, g_2\rangle$. This second pairing is called the *Peiffer lifting* (of the Peiffer commutator). Of course, if $NG_2 = 1$, then CM2 is satisfied (as for $K(\mathsf{C})$, above).

We could work with what we will call $M(G, 1)$, namely

$$\overline{\partial} : \frac{NG_1}{\partial NG_2} \to NG_0,$$

with the induced morphism and action. (As $d_0d_0 = d_0d_1$, the morphism is well defined.) This is a crossed module, but we could have divided out by less if we had wanted to. We note that $\{g_1, g_2\}$ is a product of degenerate elements, so we form, in general, the subgroup $D_n \subseteq NG_n$, generated by all degenerate elements.

Lemma 7

$$\overline{\partial} : \frac{NG_1}{\partial(NG_2 \cap D_2)} \to NG_0$$

is a crossed module. ■

This is, in fact, $M(sk_1G, 1)$, where sk_1G is the 1-skeleton of G, i.e., the subsimplicial group generated by the k-simplices for $k = 0, 1$.

The kernel of $M(G, 1)$ is $\pi_1(G)$ and the cokernel $\pi_0(G)$ and

$$\pi_1(G) \to \frac{NG_1}{\partial NG_2} \to NG_0 \to \pi_0(G)$$

represents a class $k(G) \in H^3(\pi_0(G), \pi_1(G))$. Up to a notion of 2-equivalence, $M(G, 1)$ represents the 2-type of G completely. This is an algebraic version of the result of Mac Lane and Whitehead we mentioned earlier. Once we have a bit more on cohomology, we will examine it in detail.

This use of $NG_2 \cap D_2$ and our noting that $\{g_1, g_2\}$ is a product of degenerate elements may remind you of group T-complexes and thin elements. Suppose that G is a group T-complex in the sense of our discussion at the end of the previous chapter (page 31). In a general simplicial group, the subgroups, $NG_n \cap D_n$, will not be trivial. They give measure of the extent to which homotopical information in dimension n on G depends on 'stuff' from lower dimensions., i.e., comparing G with its $(n-1)$-skeleton. (Remember that in homotopy theory, invariants such as the homotopy groups do not necessarily vanish above the dimension of the space, just recall the sphere S^2 and the subtle structure of its higher homotopy groups.)

The construction here of $M(sk_1G, 1)$ involves 'killing' the images of our possible multiple 'D-fillers' for horns, forcing uniqueness. We will see this again later.

Chapter 3

Crossed complexes

Accurate encoding of homotopy types is tricky. Chain complexes, even of G-modules, can only record certain, more or less Abelian, information. Simplicial groups, at the opposite extreme, can encode all connected homotopy types, but at the expense of such a large repetition of the essential information that makes calculation, at best, tedious and, at worst, virtually impossible. Complete information on truncated homotopy types can be stored in the cat^n-groups of Loday, [89]. We will look at these later. An intermediate model due to Blakers and Whitehead, [138], is that of a crossed complex. The algebraic and homotopy theoretic aspects of the theory of crossed complexes have been developed by Brown and Higgins, (cf. [29, 30], etc., in the bibliography and the forthcoming monograph by Brown, Higgins and Sivera, [31]) and by Baues, [14–16]. We will use them later on in several contexts.

3.1 Crossed complexes: the Definition

We will initially look at reduced crossed complexes, i.e., the group rather than the groupoid based case.

Definition: A *crossed complex*, which will be denoted C, consists of a sequence of groups and morphisms

$$\mathsf{C} : \ldots \to C_n \overset{\delta_n}{\to} C_{n-1} \overset{\delta_{n-1}}{\to} \ldots \to C_3 \overset{\delta_3}{\to} C_2 \overset{\delta_2}{\to} C_1$$

satisfying the following:
CC1) $\delta_2 : C_2 \to C_1$ is a crossed module;
CC2) each C_n, $(n > 2)$, is a left $C_1/\delta_1 C_2$-module and each δ_n, $(n > 2)$ is a morphism of left $C_1/\delta_2 C_2$-modules, (for $n = 3$, this means that δ_3 commutes with the action of C_1 and that $\delta_3(C_3) \subset C_2$ must be a $C_1/\delta_2 C_2$-module);
CC3) $\delta\delta = 0$.

The notion of a morphism of crossed complexes is clear. It is a graded collection of morphisms preserving the various structures. We thus get a category, Crs_{red} of reduced crossed complexes.

As we have that a crossed complex is a particular type of chain complex (of non-Abelian groups near the bottom), it is natural to define its homology groups in the obvious way.

Definition: If C is a crossed complex, its n^{th} *homology group* is

$$H_n(\mathsf{C}) = \frac{Ker\,\delta_n}{Im\,\delta_{n+1}}.$$

These homology groups are, of course, functors from Crs_{red} to the category of Abelian groups.

Definition: A morphism $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$ is called a *weak equivalence* if it induces isomorphisms on all homology groups.

There are good reasons for considering the homology groups of a crossed complex as being its homotopy groups. For example, if the crossed complex comes from a simplicial group then the homotopy groups of the simplicial group are the same as the homology groups of the given crossed complex (possibly shifted in dimension, depending on the notational conventions you are using).

The non-reduced version of the concept is only a bit more difficult to write down. It has C_1 as a groupoid on a set of objects C_0 with each C_k, a family of groups indexed by the elements of C_0. The axioms are very similar; see [31] for instance or many of the papers by Brown and Higgins listed in the bibliography. This gives a category, Crs, of (unrestricted) crossed complexes and morphisms between them. This category is very rich in structure. It has a tensor product structure, denoted $\mathsf{C} \otimes \mathsf{D}$ and a corresponding mapping complex construction, $\mathsf{Crs}(\mathsf{C}, \mathsf{D})$, making it into a monoidal closed category. The details are to be found in the papers and book listed above and will be recalled later when needed.

It is worth noting that this notion restricts to give us a notion of *weak equivalence* applicable to crossed modules as well.

Definition: A morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, between two crossed modules, is called a *weak equivalence* if it induces isomorphisms on π_0 and π_1, that is, on both the kernel and cokernel of the crossed modules.

The relevant reference for π_0 and π_1 is page 40.

3.1.1 Examples: crossed resolutions

As we mentioned earlier, a resolution of a group (or other object) is a model for the homotopy type represented by the group, but which usually is required to have some nice freeness properties. With crossed complexes we have some notion of homotopy around, just as with chain complexes, so we can apply that vague notion of resolution in this context as well. This will give us some neat examples of crossed complexes that are 'tuned' for use in cohomology.

A *crossed resolution* of a group G is a crossed complex, C, such that for each $n > 1$, $Im\,\delta_n = Ker\,\delta_{n-1}$ and there is an isomorphism, $C_1/\delta_2 C_2 \cong G$.

A crossed resolution can be constructed from a presentation $\mathcal{P} = (X : R)$ as follows:

Let $C(P) \to F(X)$ be the free crossed module associated with $\mathcal{P}$. We set $C_2 = C(\mathcal{P})$, $C_1 = F(X)$, $\delta_1 = \partial$. Let $\kappa(\mathcal{P}) = Ker(\partial : C(\mathcal{P}) \to F(X))$. This is the module of identities of the presentation and is a left G-module. As the category G-Mod has enough projectives, we can form a free resolution $\mathbb{P}$ of $\kappa(\mathcal{P})$. To obtain a crossed resolution of G, we join $\mathbb{P}$ to the crossed module by setting $C_n = P_{n-2}$ for $n > 3$, $\delta_n = d_{n-2}$ for $n > 3$ and the composite from P_0 to $C(P)$ for $n = 3$.

3.1.2 The standard crossed resolution

We next look at a particular case of the above, namely the standard crossed resolution of G. In this, which we will denote by $\mathsf{C}G$, we have

(i) $C_1G =$ the free group on the underlying set of G. The element corresponding to $u \in G$ will be denoted by $[u]$.

(ii) C_2G is the free crossed module over C_0G on generators, written $[u, v]$, considered as elements of the set $G \times G$, in which the map δ_1 is defined on generators by

$$\delta[u, v] = [uv]^{-1}[u][v].$$

(iii) For $n > 3$, C_nG is the free left G-module on the set G^n, but in which one has equated to zero any generator $[u_1, \ldots, u_n]$ in which some u_i is the identity element of G.

If $n > 2$, $\delta : C_{n+1}G \to C_nG$ is given by the usual formula

$$\begin{aligned}\delta[u_1, \ldots, u_{n+1}] &= {}^{[u_1]}[u_2, \ldots, u_{n+1}] \\ &\quad + \sum_{i=1}^{n} (-1)^i [u_1, \ldots, u_i u_{i+1}, \ldots, u_{n+1}] + (-1)^{n+1}[u_1, \ldots, u_n].\end{aligned}$$

For $n = 2$, $\delta : C_3G \to C_2G$ is given by

$$\delta[u, v, w] = {}^{[u]}[v, w].[u, v]^{-1}.[uv, w]^{-1}[u, vw].$$

This is the crossed analogue of the inhomogeneous bar resolution, $\mathsf{B}G$, of the group G. A groupoid version can be found in Brown-Higgins, [28], and the abstract group version in Huebschmann, [77]. In the first of these two references, it is pointed out that $\mathsf{C}G$, as constructed, is isomorphic to the crossed complex, $\underline{\pi}(BG)$, of the classifying space of G considered with its skeletal filtration.

For any filtered space, $\underline{X} = (X_n)_{n\in\mathbb{N}}$, its *fundamental crossed complex*, $\underline{\pi}(\underline{X})$, is, in general, a non-reduced crossed complex. It is defined to have

$$\underline{\pi}(\underline{X})_n = (\pi_n(X_n, X_{n-1}, a))_{a\in X_0}$$

with $\underline{\pi}(\underline{X})_1$, the fundamental groupoid $\Pi_1 X_1 X_0$, and $\underline{\pi}(\underline{X})_2$, the family, $(\pi_2(X_2, X_1, a))_{a\in X_0}$. It will only be reduced if X_0 consists just of one point.

Most of the time we will only discuss the reduced case in detail, although the non-reduced case will be needed sometimes. Following that, we will often use the notation Crs for the category of *reduced* crossed complexes unless we need the more general case. This may occasionally cause a little confusion, but it is much more convenient for most of the time.

There are two useful, but conflicting, conventions as to indexation in crossed complexes. In the topologically inspired one, the bottom group is C_1, in the simplicial and algebraic one, it is C_0. Both get used and both have good motivation. The natural indexation for the standard crossed resolution would seem to be with C_n being generated by n-tuples, i.e. the topological one. (I am not sure that all instances of the other have been avoided, so please be careful!)

G-augmented crossed complexes. Crossed resolutions of G are examples of G-augmented crossed complexes. A *G-augmented crossed complex* consists of a pair (C, φ) where C is a crossed complex and where $\varphi : C_1 \to G$ is a group homomorphism satisfying
(i) $\varphi\delta_1$ is the trivial homomorphism;
(ii) $Ker\,\varphi$ acts trivially on C_i for $i \geq 3$ and also on C_2^{Ab}.

A *morphism*

$$(\alpha, Id_G) : (\mathsf{C}, \varphi) \to (\mathsf{C}', \varphi')$$

of G-augmented crossed complexes consists of a morphism

$$\alpha : \mathsf{C} \to \mathsf{C}'$$

of crossed complexes such that $\varphi'\alpha_0 = \varphi$.

This gives a category, Crs_G, which behaves nicely with respect to change of groups, i.e. if $\varphi : G \to H$, then there are induced functors between the corresponding categories.

3.2 Crossed complexes and chain complexes: I

(Some of the proofs here are given in more detail as they are less routine and are not that available elsewhere. A source for much of this material is in the work of Brown and Higgins, [30], where these ideas were explored thoroughly for the first time; see also the treatment in [31].)

We have introduced crossed complexes where normally chain complexes of modules would have been used. We have seen earlier the bar resolution and now we have the standard crossed resolution. What is the connection between them? The answer is approximately that chain complexes form a category equivalent to a reflective subcategory of Crs. In other words, there is a canonical way of building a chain complex from a crossed one akin to the process of Abelianising a group. The resulting reflection functor sends the standard crossed resolution of a group to the bar resolution. The details involve some interesting ideas.

In chapter 2, we saw that, given a morphism $\theta : M \to N$ of modules over a group G, $\partial : M \to N \rtimes G$, given by $\partial(m) = (\theta(m), 1_G)$ is a crossed module, where $N \rtimes G$ acts on M via the projection to G. That example easily extends to a functorial construction which, from a positive chain complex, D, of G-modules, gives us a crossed complex $\Delta_G(\mathsf{D})$ with $\Delta_G(\mathsf{D})_n = D_n$ if $n > 1$ and equal to $D_1 \rtimes G$ for $n = 1$.

Lemma 8 *$\Delta_G : Ch(G{-}Mod) \to Crs_G$ is an embedding.*

Proof: That Δ_G is a functor is easy to see. It is also easy to check that it is full and faithful, that is it induces bijections,

$$Ch(G{-}Mod)(\mathsf{A}, \mathsf{B}) \to Crs_G(\Delta_G(\mathsf{A}), \Delta_G(\mathsf{B})).$$

The augmentation of $\Delta_G(A)$ is given by the projection of $A_1 \rtimes G$ onto G. ∎

We can thus turn a positive chain complex into a crossed complex. Does this functor have a left adjoint? i.e. is there a functor $\xi_G : Crs_G \to Ch(G{-}Mod)$ such that

$$Ch(G{-}Mod)(\xi_G(\mathsf{C}), \mathsf{D}) \to Crs_G(\mathsf{C}, \Delta_G(\mathsf{D}))?$$

If so it would suggest that chain complexes of G-modules are like G-augmented crossed complexes that satisfy some additional equational axioms. As an example of a similar situation think of 'Abelian groups' within 'groups' for which the inclusion has a left adjoint, namely Abelianisation $(G)^{Ab} = G/[G,G]$. Abelian groups are of course groups that satisfy the additional rule $[x,y] = 1$. Other examples of such situations are nilpotent groups of a given finite rank c. The subcategories of this general form are called *varieties* and, for instance, the study of varieties of groups is a very interesting area of group theory. Incidentally, it is possible to define various forms of cohomology modulo a variety in some sense. We will not explore that here.

We thus need to look at morphisms of crossed complexes from a crossed complex C to one of form $\Delta_G(\mathsf{D})$, and we need therefore to look at morphisms into a semidirect product. These are useful for other things, so are worth looking at in detail.

3.2.1 Semi-direct product and derivations.

Suppose that we have a diagram

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & K \rtimes G \arrow[dl, "proj"] \\ & G & \end{tikzcd}$$

where K is a G-module (written additively, so we write $g.k$ not ${}^gk$ for the action). This is like the very bottom of the situation for a morphism $f : \mathsf{C} \to \Delta_G(\mathsf{D})$.

As the codomain of f is a semidirect product, we can decompose f, as a function, in the form

$$f(h) = (f_1(h), \alpha(h)),$$

identifying its second component using the diagram. The mapping f_1 is not a homomorphism. As f is one, however, we have

$$(f_1(h_1h_2), \alpha(h_1h_2)) = f(h_1)f(h_2) = (f_1(h_1) + \alpha(h_1)f_1(h_2), \alpha(h_1h_2)),$$

i.e. f_1 satisfies

$$f_1(h_1h_2) = f_1(h_1) + \alpha(h_1)f_1(h_2)$$

for all $h_1, h_2 \in H$.

3.2.2 Derivations and derived modules.

We will use the identification of G-modules for a group G with modules over the group ring, $\mathbb{Z}[G]$, of G. Recall that this ring is obtained from the free Abelian group on the set G by defining a multiplication extending linearly that of G itself. (Formally if, for the moment, we denote by e_g, the generator corresponding to $g \in G$, then an arbitrary element of $\mathbb{Z}[G]$ can be written as $\sum_{g\in G} n_g e_g$ where the n_g are integers and only finitely many of them are non-zero. The multiplication is by 'convolution' product, that is,

$$\Big(\sum_{g\in G} n_g e_g\Big)\Big(\sum_{g\in G} m_g e_g\Big) = \sum_{g\in G}\Big(\sum_{g_1\in G} n_{g_1} m_{g_1^{-1}g} e_g\Big).$$

Sometimes, later on, we will need other coefficients that $\mathbb{Z}$ in which case it is appropriate to use the term 'group algebra' of G, over that ring of coefficients.

We will also need the augmentation, $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$, given by $\varepsilon(\sum_{g\in G} n_g e_g) = \sum_{g\in G} n_g$ and its kernel $I(G)$, known as the *augmentation ideal*.

Definitions: Let $\varphi : G \to H$ be a homomorphism of groups. A φ-*derivation*

$$\partial : G \to M$$

from G to a left $\mathbb{Z}[H]$-module, M, is a mapping from G to M, which satisfies the equation

$$\partial(g_1 g_2) = \partial(g_1) + \varphi(g_1)\partial(g_2)$$

for all $g_1,\, g_2 \in G$.

Such φ-derivations are really all derived from a universal one.

Definition: A *derived module* for φ consists of a left $\mathbb{Z}[H]$-module, D_φ, and a φ-derivation, $\partial_\varphi : G \to D_\varphi$ with the following universal property:

Given any left $\mathbb{Z}[H]$-module, M, and a φ-derivation $\partial : G \to M$, there is a unique morphism

$$\beta : D_\varphi \to M$$

of $\mathbb{Z}[H]$-modules such that $\beta\partial_\varphi = \partial$.

The derivation ∂_φ is called the *universal* φ *derivation*.

The set of all φ-derivations from G to M has a natural Abelian group structure. We denote this set by $Der_\varphi(G, M)$. This gives a functor from *H-Mod* to *Ab*, the category of Abelian groups. If $(D_\varphi, \partial_\varphi)$ exists, then it sets up a natural isomorphism

$$Der_\varphi(G, M) \cong H{-}Mod(D_\varphi, M),$$

i.e., $(D_\varphi, \partial_\varphi)$ represents the φ-derivation functor.

3.2.3 Existence

The treatment of derived modules that is found in Crowell's paper, [51], provides a basis for what follows. In particular it indicates how to prove the existence of $(D_\varphi, \partial_\varphi)$ for any φ.

Form a $\mathbb{Z}[H]$-module, D, by taking the free left $\mathbb{Z}[H]$-module, $\mathbb{Z}[H]^{(X)}$, on a set of generators, $X = \{\partial g : g \in G\}$. Within $\mathbb{Z}[H]^{(X)}$ form the submodule, Y, generated by the elements

$$\partial(g_1 g_2) - \partial(g_1) - \varphi(g_1)\partial(g_2).$$

Let $D = \mathbb{Z}[H]^{(X)}/Y$ and define $d : G \to D$ to be the composite:

$$G \xrightarrow{\eta} \mathbb{Z}[H]^{(X)} \xrightarrow{quotient} D,$$

where η is "inclusion of the generators", $\eta(g) = \partial g$, thus d, by construction, will be a φ-derivation. The universal property is easily checked and hence $(D_\varphi, \partial_\varphi)$ exists.

We will later on construct $(D_\varphi, \partial_\varphi)$ in a different way which provides a more amenable description of D_φ, namely as a tensor product. As a first step towards this description, we shall give a simple description of D_G, that is, the derived module of the identity morphism of G. More precisely we shall identify (D_G, ∂_G) as being $(I(G), \partial)$, where, as above, $I(G)$ is the augmentation ideal of $\mathbb{Z}[G]$ and $\partial : G \to I(G)$ is the usual map, $\partial(g) = g - 1$.

Our earlier observations give us the following useful result:

Lemma 9 *If G is a group and M is a G-module, then there is an isomorphism*

$$Der_G(G, M) \to Hom/G(G, M \rtimes G)$$

where $Hom/G(G, M \rtimes G)$ is the set of homomorphisms from G to $M \rtimes G$ over G, i.e., $\theta : G \to M \rtimes G$ such that for each $g \in G$, $\theta(g) = (g, \theta'(g))$ for some $\theta'(g) \in M$. ■

3.2.4 Derivation modules and augmentation ideals

Proposition 5 *The derivation module D_G is isomorphic to $I(G) = Ker(\mathbb{Z}[G] \to \mathbb{Z})$. The universal derivation is*

$$d_G : G \to I(G)$$

given by $d_G(g) = g - 1$.

Proof:

We introduce the notation $f_\delta : I(G) \to M$ for the $\mathbb{Z}[G]$-module morphism corresponding to a derivation

$$\delta : G \to M.$$

The factorisation $f_\delta d_G = \delta$ implies that f_δ must be defined by $f_\delta(g - 1) = \delta(g)$. That this works follows from the fact that $I(G)$, as an Abelian group, is free on the set $\{g - 1 : g \in G\}$ and that the relations in $I(G)$ are generated by those of the form

$$g_1(g_2 - 1) = (g_1 g_2 - 1) - (g_1 - 1).$$

■

We note a result on the augmentation ideal construction that is not commonly found in the literature.

The proof is easy and so will be omitted.

Lemma 10 *Given groups G and H in $\mathcal{C}$ and a commutative diagram*

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\psi"'] & M \arrow[d, "\varphi"] \\ H \arrow[r, "\delta'"'] & N \end{tikzcd} \qquad (*)$$

where δ, δ' are derivations, M is a left $\mathbb{Z}[G]$-module, N is a left $\mathbb{Z}[H]$-module and φ is a module map over ψ, i.e., $\varphi(g.m) = \psi(g)\varphi(m)$ for $g \in G$, $m \in M$. Then the corresponding diagram

$$\begin{tikzcd} I(G) \arrow[r, "f_\delta"] \arrow[d, "\psi"'] & M \arrow[d, "\varphi"] \\ I(H) \arrow[r, "f_{\delta'}"'] & N \end{tikzcd} \qquad (**)$$

is commutative. ∎

The earlier proposition has the following corollaries:

Corollary 1 *The subset $Im\, d_G = \{g - 1 : g \in G\} \subset I(G)$ generates $I(G)$ as a $\mathbb{Z}[G]$-module. Moreover the relations between these generators are generated by those of the form*

$$(g_1 g_2 - 1) - (g_1 - 1) - g_1(g_2 - 1).$$

∎

It is useful to have also the following reformulation of the above results stated explicitly.

Corollary 2 *There is a natural isomorphism*

$$Der_G(G, M) \cong G{-}Mod(I(G), M).$$

∎

3.2.5 Generation of $I(G)$.

The first of these two corollaries raises the question as to whether, if $X \subset G$ generates G, does the set $G_X = \{x - 1 : x \in X\}$ generate $I(G)$ as a $\mathbb{Z}[G]$-module.

Proposition 6 *If X generates G, then G_X generates $I(G)$.*

Proof: We know $I(G)$ is generated by the $g - 1$s for $g \in G$. If g is expressible as a word of length n in the generators X then we can write $g - 1$ as a $\mathbb{Z}[G]$-linear combination of terms of the form $x-1$ in an obvious way. (If $g = w.x$ with w of lesser length than that of g, $g-1 = w-1+w(x-1)$, so use induction on the length of the expression for g in terms of the generators.) ∎

When G is free: If G is free, say, $G \cong F(X)$, i.e., is free on the set X, we can say more.

Proposition 7 *If $G \cong F(X)$ is the free group on the set X, then the set $\{x - 1 : x \in X\}$ freely generates $I(G)$ as a $\mathbb{Z}[G]$-module.*

Proof: (We will write F for $F(X)$.) The easiest proof would seem to be to check the universal property of derived modules for the function $\delta : F \to \mathbb{Z}[G]^{(X)}$, given on generators by

$$\delta(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}$$

then extended using the derivation rule to all of F using induction. This uses essentially that each element of F has a *unique* expression as a reduced word in the generators, X.

Suppose then that we have a derivation $\partial : F \to M$, define $\overline{\partial} : \mathbb{Z}[G]^{(X)} \to M$ by $\overline{\partial}(e_x) = \partial(x)$, extending linearly. Since by construction $\overline{\partial}\delta = \partial$ and is the unique such homomorphism, we are home. ■

Note: In both these proofs we are thinking of the elements of the free module on X as being functions from X to the group ring, these functions being of 'finite support', i.e. being non-zero on only a finite number of elements of X. This can cause some complications if X is infinite or has some topology as it will in some contexts. The *idea* of the proof will usually go across to that situation but details have to change. (A situation in which this happens is in profinite group theory where the derivations have to be continuous for the profinite topology on the group, see [111].)

3.2.6 (D_φ, d_φ), the general case.

We can now return to the identification of (D_φ, d_φ) in the general case.

Proposition 8 *If $\varphi : G \to H$ is a homomorphism of groups, then $D_\varphi \cong \mathbb{Z}[H] \otimes_G I(G)$, the tensor product of $\mathbb{Z}[H]$ and $I(G)$ over G.*

Proof: If M is a $\mathbb{Z}[H]$-module, we will write $\varphi^*(M)$ for the restricted $\mathbb{Z}[G]$-module, i.e. M with G-action given by $g.m := \varphi(g).m$. Recall that the functor φ^* has a left adjoint given by sending a G-module, N to $\mathbb{Z}[H] \otimes_G N$, i.e. take the tensor of Abelian groups, $\mathbb{Z}[H] \otimes N$ and divide out by $x \otimes g.n \equiv x\varphi(g) \otimes n$.

With this notation we have a chain of natural isomorphisms,

$$\begin{aligned} Der_\varphi(G, M) &\cong Der_G(G, \varphi^*(M)) \\ &\cong G{-}Mod(I(G), \varphi^*(M)) \\ &\cong H{-}Mod(\mathbb{Z}[H] \otimes_G I(G), M), \end{aligned}$$

so by universality,

$$D_\varphi \cong \mathbb{Z}[H] \otimes_G I(G),$$

as required. ■

3.2.7 D_φ for $\varphi : F(X) \to G$.

The above will be particularly useful when φ is the "co-unit" map, $F(X) \to G$, for X a set that generates G. We could, for instance, take $X = G$ as a set, and φ to be the usual natural epimorphism.

In fact we have the following:

Corollary 3 *Let $\varphi : F(X) \to G$ be an epimorphism of groups, then there is an isomorphism*

$$D_\varphi \cong \mathbb{Z}[G]^{(X)}$$

of $\mathbb{Z}[G]$-modules. In this isomorphism, the generator ∂_x, of D_φ corresponding to $x \in X$, satisfies

$$d_\varphi(x) = \partial_x$$

for all $x \in X$. ■

(You should check that you see how this follows from our earlier results.)

3.3 Associated module sequences

3.3.1 Homological background

Given an exact sequence

$$1 \to K \to L \to Q \to 1$$

of abstract groups, then it is a standard result from homological algebra that there is an associated exact sequence of modules,

$$0 \to K^{Ab} \to \mathbb{Z}[Q] \otimes_L I(L) \to I(Q) \to 0.$$

There are several different proofs of this. Homological proofs give this as a simple consequence of the Tor^L-sequence corresponding to the exact sequence

$$0 \to I(L) \to \mathbb{Z}[L] \to \mathbb{Z} \to 0$$

together with a calculation of $Tor_1^L(\mathbb{Z}[Q], \mathbb{Z})$, but we are not assuming that much knowledge of standard homological algebra. That homological proof also, to some extent, hides what is happening at the 'elementary' level, in both the sense of 'simple' and also that of'what happens to the 'elements' of the groups and modules concerned.

The second type of proof is more directly algebraic and has the advantage that it accentuates various universal properties of the sequence. The most thorough treatment of this would seem to be by Crowell, [51], for the discrete case. We outline it below.

3.3.2 The exact sequence.

Before we start on the discussion of the exact sequence, it will be useful to have at our disposal some elementary results on Abelianisation of the groups in a crossed module. Here we actually only need them for normal subgroups but we will need it shortly anyway in the more general form. Suppose that (C, P, ∂) is a crossed module, and we will set $A = Ker\partial$ with its module structure that we looked at before, and $N = \partial C$, so A is a P/N-module.

Lemma 11 *The Abelianisation of C has a natural $\mathbb{Z}[P/N]$-module structure on it.*

Proof: First we should point out that by "Abelianisation" we mean $C^{Ab} = C/[C, C]$, which is, of course, Abelian and it suffices to prove that N acts trivially on C^{Ab}, since P already acts in a natural way. However, if $n \in N$, and $\partial c = n$, then for any $c' \in C$, we have that ${}^n c' = {}^{\partial c} c' = cc'c^{-1}$, hence ${}^n c'(c')^{-1} \in [C, C]$ or equivalently

$${}^n(c'[C, C]) = c'[C, C],$$

so N does indeed act trivially on C^{Ab}. ■

Of course N^{Ab} also has the structure of a $\mathbb{Z}[P/N]$-module and thus a crossed module gives one three P/N-modules. These three are linked as shown by the following proposition.

Proposition 9 *Let (C, P, ∂) be a crossed module. Then the induced morphisms*

$$A \to C^{Ab} \to N^{Ab} \to 0$$

form an exact sequence of $\mathbb{Z}[P/N]$-modules.

Proof: It is clear that the sequence

$$1 \to A \to C \to N \to 1$$

is exact and that the induced homomorphism from C^{Ab} to N^{Ab} is an epimorphism. Since the composite homomorphism from A to N is trivial, A is mapped into $Ker(C^{Ab} \to N^{Ab})$ by the composite $A \to C \to C^{Ab}$. It is easily checked that this is onto and hence the sequence is exact as claimed. ■

Now for the main exact sequence result here:

Proposition 10 *Let*

$$1 \to K \xrightarrow{\varphi} L \xrightarrow{\psi} Q \to 1$$

be an exact sequence of groups and homomorphisms. Then there is an exact sequence

$$0 \to K^{Ab} \xrightarrow{\tilde{\varphi}} \mathbb{Z}[Q]\otimes_L I(L) \xrightarrow{\tilde{\psi}} I(Q) \to 0$$

of $\mathbb{Z}[Q]$*-modules.*

Proof: By the universal property of D_ψ, there is a unique morphism

$$\tilde{\psi} : D_\psi \to I(Q)$$

such that $\tilde{\psi}\partial_\psi = I(\psi)\partial_L$.

Let $\delta : K \to K^{Ab} = K/[K, K]$ be the canonical Abelianising morphism. We note that $\partial_\psi\varphi : K \to D_\psi$ is a homomorphism (since

$$\begin{aligned} \partial_\psi\varphi(k_1k_2) &= \partial_\psi\varphi(k_1) + \psi\varphi(k_1)\partial_\psi\varphi(k_2) \\ &= \partial_\psi\varphi(k_1) + \partial_\psi\varphi(k_2),) \end{aligned}$$

so let $\tilde{\varphi} : K^{Ab} \to D_\psi$ be the unique morphism satisfying $\tilde{\varphi}\delta = \partial_\psi\varphi$ with K^{Ab} having its natural $\mathbb{Z}[Q]$-module structure.

That the composite $\tilde{\psi}\tilde{\varphi} = 0$ follows easily from $\psi\varphi = 0$. Since D_ψ is generated by symbols $d\ell$ and $\tilde{\psi}(d\ell) = \psi(\ell) - 1$, it follows that $\tilde{\psi}$ is onto. We next turn to "$Ker\ \tilde{\psi} \subseteq Im\ \tilde{\varphi}$".

If we can prove $\alpha : D_\psi \to I(Q)$ is the cokernel of $\tilde{\varphi}$, then we will have checked this inclusion and incidentally will have reproved that $\tilde{\psi}$ is onto.

Now let $D_\psi \to C$ be any morphism such that $\alpha\tilde{\varphi} = 0$. Consider the diagram

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\delta"'] & L \arrow[r, "\psi"] \arrow[d, "\partial_\psi"] & Q \arrow[d, "\partial_Q"] \\ K^{Ab} \arrow[r, "\tilde{\varphi}"] & D_\psi \arrow[r, "\tilde{\psi}"] \arrow[dr, "\alpha"'] & c(Q) \\ & & C \end{tikzcd}$$

The composite $\alpha\partial_\psi$ vanishes on the image of φ since $\alpha\partial_\psi\varphi = \alpha\tilde{\varphi}\delta$ and $\alpha\tilde{\varphi}$ is assumed zero. Define $d : Q \to C$ by $d(q) = \alpha\partial_\psi(\ell)$ for $\ell \in L$ such that $\psi(\ell) = q$. As $\alpha\partial_\psi$ vanishes on $Im\ \varphi$, this is well defined and

$$\begin{aligned} d(q_1q_2) &= \alpha\partial_\psi(\ell_1\ell_2) \\ &= \alpha\partial_\psi(\ell_1) + \alpha(\psi(\ell_1)\partial_\psi(\ell_2)) \\ &= d(q_1) + q_1d(q_2) \end{aligned}$$

so d factors as $\bar{\alpha}\partial_Q$ in a unique way with $\bar{\alpha} : I(Q) \to C$. It remains to prove that $\alpha = \tilde{\psi}$, but

$$\begin{aligned} \tilde{\psi}\partial_\psi &= I_C(\psi)\partial_L \\ &= \partial_Q\psi \end{aligned}$$

by the naturality of ∂. Now finally note that $\bar{\alpha}\partial_Q = d$ and $d\psi = \alpha\partial_\psi$ to conclude that $\tilde{\psi}\partial_\psi$ and $\alpha\partial_\psi$ are equal. Equality of α and $\bar{\alpha}\tilde{\psi}$ then follows by the uniqueness clause of the universal property of (D_ψ, ∂_ψ).

Next we need to check that $K^{Ab} \to D_\psi$ is a monomorphism. To do this we use the fact that there is a transversal, $s : Q \to L$, satisfying $s(1) = 1$. This means that, following Crowell, [51] p. 224, we can for each $\ell \in L$, $q \in Q$, find an element $q \times \ell$ uniquely determined by the equation

$$\varphi(q \times \ell)) = s(q)\ell s(q\psi(\ell))^{-1},$$

which, of course, defines a function from $Q \times L$ to K. Crowell's lemma 4.5 then shows

$$q \times \ell_1\ell_2 = (q \times \ell_1)(q\psi(\ell_1) \times \ell_2) \ \text{ for } \ \ell_1, \ell_2 \in L.$$

Now let $M = \mathbb{Z}[Q]^{(X)}$, with $X = \{\partial\ell : \ell \in L\}$, so that there is an exact sequence

$$M \to D_\psi \to 0.$$

The underlying group of $\mathbb{Z}[Q]$ is the free Abelian group on the underlying set of Q. Similarly M, above, has, as underlying group, the free Abelian group on the set $Q \times X$.

Define a map $\tau : M \to K^{Ab}$ of Abelian groups by

$$\tau(a, \partial\ell) = \delta(q \times \ell).$$

We check that if $p(m) = 0$, then $\tau(m) = 0$. Since $Ker\, p$ is generated as a $\mathbb{Z}[Q]$-module by elements of the form

$$\partial(\ell_1\ell_2) - \partial\ell_1 - \psi(\ell_1)\partial\ell_2,$$

it follows that as an Abelian group, $Ker\, p$ is generated by the elements

$$(q, \partial(\ell_1\ell_2)) - (q, \partial\ell_1) - (q\psi(\ell_1), \partial\ell_2).$$

We claim that τ is zero on these elements; in fact

$$\begin{aligned} \tau(q, \partial(\ell_1\ell_2)) &= \delta(q \times (\ell_1\ell_2)) \\ &= \delta(q \times \ell_1) + \delta(q\psi(\ell_1) \times \ell_2) \\ &= \tau(q, \ell_1) + \tau(q\psi(\ell_1), \ell_2). \end{aligned}$$

Thus τ induces a map $\eta : D_\psi \to K^{Ab}$ of Abelian groups.

Finally we check $\eta\tilde{\varphi} =$ identity, so that $\tilde{\varphi}$ is a monomorphism: let $b \in K^{Ab}$, $k \in K$ be such that $\delta(k) = b$, then

$$\begin{aligned} \eta\tilde{\varphi}(b) &= \eta\tilde{\varphi}\delta(k) \\ &= \eta\partial_\psi(k) \\ &= \delta(1 \times \varphi(k)), \end{aligned}$$

but $1 \times \varphi(k)$ is uniquely determined by

$$\varphi(1 \times \varphi(k)) = s(1)\varphi(k)s(1\psi\varphi(k))^{-1} = \varphi(k),$$

since $s(1) = 1$, hence $1 \times \varphi(k) = k$ and $\eta\tilde{\varphi}(b) = \delta(k) = b$ as required. ∎

A discussion of the way in which this result interacts with the theory of covering spaces can be found in Crowell's paper already cited. We will very shortly see the connection of this module sequence with the Jacobian matrix of a group presentation and the Fox free differential calculus. It is this latter connection which suggests that we need more or less explicit formulae for the maps $\tilde{\varphi}$ and $\tilde{\psi}$ and hence requires that Crowell's detailed proof be used, not the slicker homological proof.

3.3.3 Reidemeister-Fox derivatives and Jacobian matrices

At various points, we will refer to Reidemeister-Fox derivatives as developed by Fox in a series of articles, see [63], and also summarised in Crowell and Fox, [52]. We will call these derivatives *Fox derivatives.*

Suppose G is a group and M a G-module and let $\delta : G \to M$ be a derivation, (so $\delta(g_1g_2) = \delta(g_1) + g_1\delta(g_2)$ for all $g_1, g_2 \in G$), then, for calculations, the following lemma is very valuable, although very simple to prove.

Lemma 12 *If $\delta : G \to M$ is a derivation, then*
(i) $\delta(1_G) = 0$;
(ii) $\delta(g^{-1}) = -g^{-1}\delta(g)$ for all $g \in G$;
(iii) for any $g \in G$ and $n \geq 1$,

$$\delta(g^n) = (\sum_{k=0}^{n-1} g^k)\delta(g).$$

Proof: As was said, these are easy to prove.
$\delta(g) = \delta(1g) + 1\delta(g)$, so $\delta(1) = 0$, and hence (i); then

$$\delta(1) = \delta(g^{-1}g) = \delta(g^{-1}) + g^{-1}\delta(g)$$

to get (ii), and finally induction to get (iii). ∎

The Fox derivatives are derivations taking values in the group ring as a left module over itself. They are defined for $G = F(X)$, the free group on a set X. (We usually write F for $F(X)$ in what follows.)

Definition: For each $x \in X$, let
$$\frac{\partial}{\partial x} : F \to \mathbb{Z}F$$
be defined by
(i) for $y \in X$,
$$\frac{\partial y}{\partial x} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}$$
(ii) for any words, $w_1, w_2 \in F$,
$$\frac{\partial}{\partial x}(w_1 w_2) = \frac{\partial}{\partial x} w_1 + w_1 \frac{\partial}{\partial x} w_2.$$
Of course, a routine proof shows that the derivation property in (ii) defines $\frac{\partial w}{\partial x}$ for any $w \in F$.
This derivation, $\frac{\partial}{\partial x}$, will be called the *Fox derivative with respect to the generator x*.

Example: Let $X = \{u, v\}$, with $r \equiv uvuv^{-1}u^{-1}v^{-1} \in F = F(u, v)$, then
$$\begin{aligned} \frac{\partial r}{\partial u} &= 1 + uv - uvuv^{-1}u^{-1}, \\ \frac{\partial r}{\partial v} &= u - uvuv^{-1} - uvuv^{-1}u^{-1}v^{-1}. \end{aligned}$$
This relation is the typical braid group relation, here in Br_3, and we will come back to these simple calculations later.

It is often useful to extend a derivation $\delta : G \to M$ to a linear map from $\mathbb{Z}G$ to M by the simple rule that $\delta(g + h) = \delta(g) + \delta(h)$.
We have
$$Der(F, \mathbb{Z}F) \cong F\!-\!Mod(IF, \mathbb{Z}F),$$
and that
$$IF \cong \mathbb{Z}F^{(X)},$$
with the isomorphism matching each generating $x - 1$ with e_x, the basis element labelled by $x \in X$. (The universal derivation then sends x to e_x.)
For each given x, we thus obtain a morphism of F-modules:
$$d_x : \mathbb{Z}F^{(X)} \to \mathbb{Z}F$$
with
$$\begin{aligned} d_x(e_y) &= 1 \quad \text{if } y = x \\ d_x(e_y) &= 0 \quad \text{if } y \neq x, \end{aligned}$$
i.e., the 'projection onto the x^{th}-factor' or 'evaluation at $x \in X$' depending on the viewpoint taken of the elements of the free module, $\mathbb{Z}F^{(X)}$.
Suppose now that we have a group presentation, $\mathcal{P} = (X : R)$, of a group, G. Then we have a short exact sequence of groups
$$1 \to N \stackrel{\varphi}{\to} F \stackrel{\gamma}{\to} G \to 1,$$

where $N = N(R)$, $F = F(X)$, i.e., N is the normal closure of R in the free group F. We also have a free crossed module,

$$C \xrightarrow{\partial} F,$$

constructed from the presentation and hence, two short exact sequences of G-modules with $\kappa(\mathcal{P}) = Ker\,\partial$, the module of identities of $\mathcal{P}$,

$$0 \to \kappa(\mathcal{P}) \to C^{Ab} \to N^{Ab} \to 0,$$

and also

$$0 \to N^{Ab} \xrightarrow{\tilde{\varphi}} IF \otimes_F \mathbb{Z}G \to IG \to 0.$$

We note that the first of these is exact because N is a free group, (see Proposition 12, which will be proved shortly), further

$$C^{Ab} \cong \mathbb{Z}G^{(R)},$$

(the proof is left to you to manufacture from earlier results), and the map from this to N^{Ab} in the first sequence sends the generator e_r to $r[N, N]$.

We next revisit the derivation of the associated exact sequence (Proposition 10, page 65) in some detail to see what $\tilde{\varphi}$ does to $r[N, N]$. We have $\tilde{\varphi}(r[N, N]) = \partial_\gamma \varphi(r) = \partial_\gamma(r)$, considering r now as an element of F, and by Corollary 3, on identifying D_γ with $\mathbb{Z}G^{(X)}$ using the isomorphism between IF and $\mathbb{Z}F^{(X)}$, we can identify $\partial_\gamma(x) = e_x$. We are thus left to determine $\partial_\gamma(r)$ in terms of the $\partial_\gamma(x)$, i.e., the e_x. The following lemma does the job for us.

Lemma 13 *Let $\delta : F \to M$ be a derivation and $w \in F$, then*

$$\delta w = \sum_{x \in X} \frac{\partial w}{\partial x} \delta x.$$

Proof: By induction on the length of w. ■

In particular we thus can calculate

$$\partial_\gamma(r) = \sum \frac{\partial r}{\partial x} e_x.$$

Tensoring with $\mathbb{Z}G$, we get

$$\tilde{\varphi}(r[N, N]) = \sum \frac{\partial r}{\partial x} e_x \otimes 1.$$

There is one final step to get this into a usable form:

From the quotient map $\gamma : F \to G$, we, of course, get an induced ring homomorphism, $\gamma : \mathbb{Z}F \to \mathbb{Z}G$, and hence we have elements $\gamma(\frac{\partial r}{\partial x}) \in \mathbb{Z}G$. Of course,

$$\frac{\partial r}{\partial x} e_x \otimes 1 = e_x \otimes \gamma(\frac{\partial r}{\partial x}),$$

so we have, on tidying up notation just a little:

Proposition 11 *The composite map*

$$\mathbb{Z}G^{(R)} \to N^{Ab} \to \mathbb{Z}G^{(X)}$$

sends e_r to $\sum \gamma(\frac{\partial r}{\partial x}) e_x$ and so has a matrix representation given by $J_{\mathcal{P}} = \big(\gamma(\frac{\partial r_i}{\partial x_j})\big)$. ■

Definition: The *Jacobian matrix* of a group presentation, $\mathcal{P} = (X : R)$ of a group G is

$$J_{\mathcal{P}} = \Big(\gamma(\frac{\partial r_i}{\partial x_j})\Big),$$

in the above notation.

The application of γ to the matrix of Fox derivatives simplifies expressions considerable in the matrix. The usual case of this is if a relator has the form rs^{-1}, then we get

$$\frac{\partial rs^{-1}}{\partial x} = \frac{\partial r}{\partial x} - rs^{-1}\frac{\partial s}{\partial x}$$

and if r or s is quite long this looks moderately horrible to work out! However applying γ to the answer, the term rs^{-1} in the second of the two terms becomes 1. We can actually think of this as replacing rs^{-1} by $r - s$ when working out the Jacobian matrix.

Example: Br_3 revisited. We have $r \equiv uvuv^{-1}u^{-1}v^{-1}$, which has the form $(uvu)(vuv)^{-1}$. This then gives

$$\gamma(\frac{\partial r}{\partial u}) = 1 + uv - v \quad \text{and} \quad \gamma(\frac{\partial r}{\partial v}) = u - 1 - vu,$$

abusing notation to ignore the difference between u, v in $F(u, v)$ and the generating u, v in Br_3.

Homological 2-syzygies: In general we obtain a truncated chain complex:

$$\mathbb{Z}G^{(R)} \overset{d_2}{\to} \mathbb{Z}G^{(X)} \overset{d_1}{\to} \mathbb{Z}G \overset{d_0}{\to} \mathbb{Z} \to 0,$$

with d_2 given by the Jacobian matrix of the presentation, and d_1 sending generator e^1_x to $1 - x$, so $Im\, d_1$ is the augmentation ideal of $\mathbb{Z}G$.

Definition: A *homological 2-syzygy* is an element in $Ker\, d_2$.

A homological 2-syzygy is thus an element to be killed when building the third level of a resolution of G. What are the links between homotopical and homological syzygies? Brown and Huebschmann, [32], show they are isomorphic, as $Ker\, d_2$ is isomorphic to the module of identities. We will examine this result in more detail shortly.

Extended example: Homological Syzygies for the braid group presentations: The Artin braid group, Br_{n+1}, defined using $n+1$ strands is given by

- generators: y_i, $i = 1, \ldots, n$;
- relations: $r_{ij} \equiv y_i y_j y_i^{-1} y_j^{-1}$ for $i + 1 < j$;
  $r_{ii+1} \equiv y_i y_{i+1} y_i y_{i+1}^{-1} y_i^{-1} y_{i+1}^{-1}$ for $1 \leq i < n$.

We will look at such groups only for small values of n.

By default, Br_2 has one generator and no relations, so is infinite cyclic.

The group Br_3: (We will simplify notation writing $u = y_1$, $v = y_2$.)

This then has presentation $\mathcal{P} = (u, v : r \equiv uvuv^{-1}u^{-1}v^{-1})$. It is also the 'trefoil group', i.e., the fundamental group of the complement of a trefoil knot. If we construct $X(2) = K(\mathcal{P})$, this is already a $K(Br_3, 1)$ space, having a trivial π_2. There are no higher syzygies.

We have all the calculation for working with homological syzygies here. The key part of the complex is the Jacobian matrix as that determines d_2:

$$d_2 = \begin{pmatrix} 1 + uv - v & u - 1 - vu \end{pmatrix}.$$

This has trivial kernel, but, in fact, that comes most easily from the identification with homotopical syzygies.

The group Br_4: simplifying notation as before, we have generators u, v, w and relations

$$\begin{array}{rcl} r_u & \equiv & vwvw^{-1}v^{-1}w^{-1}, \\ r_v & \equiv & uwu^{-1}w^{-1}, \\ r_w & \equiv & uvuv^{-1}u^{-1}v^{-1}. \end{array}$$

The 1-syzygies are made up of hexagons for r_u and r_w and a square for r_v. There is a fairly obvious way of fitting together squares and hexagons, namely as a permutohedron, and there is a labelling of such that gives a homotopical 2-syzygy.

The presentation yields a truncated chain complex with d_2

$$\mathbb{Z}G^{(r_u, r_v, r_w)} \xrightarrow{d_2} \mathbb{Z}G^{(u,v,w)}$$

with

$$d_2 = \begin{pmatrix} 0 & 1 + vw - w & v - 1 - wv \\ 1 - w & 0 & u - 1 \\ 1 + uv - v & u - 1 - vu & 0 \end{pmatrix}$$

and Loday, [90], has calculated that for the permutohedral 2-syzygy, s, one gets another term of the resolution, $\mathbb{Z}G^{(s)}$, and a $d_3 : \mathbb{Z}G^{(s)} \to \mathbb{Z}G^{(r_u, r_v, r_w)}$ given by

$$d_3 = \begin{pmatrix} 1 + vu - u - wuv & v - vwu - 1 - uv - vuwv & 1 + vw - w - uvw \end{pmatrix}.$$

For more on methods of working with these syzygies, have a look at Loday's paper, [90], and some of the references that you will find there.

3.4 Crossed complexes and chain complexes: II

(The source for the material and ideas in this section is once again [30].)

3.4.1 The reflection from Crs to chain complexes

It is now time to return to the construction of a left adjoint for Δ_G.

Theorem 3 *(Brown-Higgins, [30] in a slightly more general form.) The functor, Δ_G, has a left adjoint.*

Proof: We construct the left adjoint explicitly as follows:

Let $f. : (\mathsf{C}, \varphi) \to \Delta_G(M.)$ be a morphism in Crs_G, then we have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\varphi"] \arrow[d, "f_0"] & G \arrow[d, "Id_G"] \\
\cdots \arrow[r] & M_2 \arrow[r, "\delta_2"] & M_1 \arrow[r, "\delta_1"] & M_0 \rtimes G \arrow[r, "pr_G"] & G
\end{tikzcd}$$

Since the right hand square commutes, f_0 is given by some formula

$$f_0(c) = (\partial(c), \varphi(c)),$$

where $\partial : C_0 \to M_0$ is a φ-derivation. Thus $\partial = \tilde{f}_0 \partial_\varphi$ for a unique G-module morphism, $\tilde{f}_0 : D_\varphi \to M_0$, and f_0 factors as

$$C_0 \stackrel{\bar{\varphi}}{\to} D_\varphi \rtimes G \stackrel{\tilde{f}_0 \rtimes G}{\to} M_0 \rtimes G,$$

where $\bar{\varphi}(c) = (\partial_\varphi(c), \varphi(c))$.

The map $\partial_\varphi \delta_1 : C_1 \to D_\varphi$ is a homomorphism since

$$\begin{aligned}
\partial_\varphi \delta_1(c_1 c_2) &= \partial_\varphi \partial_1(c_1) + \varphi\partial_1(c_1)\partial_\varphi\partial_1(c_2) \\
&= \partial_\varphi \partial_1(c_1) + \partial_\varphi \partial_1(c_2),
\end{aligned}$$

$\varphi\partial_1$ being trivial (because (C, φ) is G-augmented). We thus obtain a map $d : C_1^{Ab} \to D_\varphi$ given by $d(c[C,C]) = \partial_\varphi \partial_1(c)$ for $c \in C_1$. As we observed earlier the Abelian group C_1^{Ab} has a natural $\mathbb{Z}[G]$-module structure making d a G-module morphism.

Similarly there is a unique G-module morphism,

$$\tilde{f}_1 : C_1^{Ab} \to M_1,$$

satisfying

$$\tilde{f}_1(c[C,C]) = f_1(c).$$

Since for $c \in C_1$,

$$(d_1 \tilde{f}_1(c), 1) = f_0(\delta_1 c) = (\tilde{f}_0 \partial_\varphi(\delta_1 c_1), 1),$$

we have that the diagram

$$\begin{tikzcd}
C_1^{Ab} \arrow[r, "\tilde{f}_1"] \arrow[d, "d"'] & M_1 \arrow[d, "d_1"] \\
D_\varphi \arrow[r, "\tilde{f}_0"] & M_0
\end{tikzcd}$$

commutes.

We also note that since $\delta_2 : C_2 \to C_1$ maps into $Ker\ \delta_1$, the composite

$$C_2 \stackrel{\delta_2}{\to} C_1 \stackrel{\text{can}}{\to} C_1^{Ab} \stackrel{d}{\to} D_\varphi,$$

being given by $d(\delta_2(c)[C,C] = \partial_\varphi \delta_1 \delta_2(c)$, is trivial and that $\tilde{f}_1 \delta_2(c[C,C]) = f_1\delta_2(c) = d_2 f_2(c)$, thus we can define $\xi = \xi_G(\mathsf{C}, \varphi)$ by

$$\begin{aligned}
\xi_n &= C_n \text{ if } n \geq 2 \\
\xi_1 &= C_1^{Ab}, \\
\xi_0 &= D_\varphi,
\end{aligned}$$

the differentials being as constructed. We note that as $Ker\ \varphi$ acts trivially on all C_n for $n \geq 2$, all the C_n have $\mathbb{Z}[G]$-module structures.

That ξ_G gives a functor

$$Crs \to Ch(G{-}Mod)$$

is now easy to check using the uniqueness clauses in the universal properties of D_φ and Abelianisation. Again uniqueness guarantees that the process "f goes to $\tilde{f}$" gives a natural isomorphism

$$Ch(G-Mod)(\xi_G(\mathsf{C},\varphi),\mathsf{M}) \cong Crs_G((\mathsf{C},\varphi),\Delta_G(\mathsf{M}))$$

as required. ■

It is relatively easy to extend the above natural isomorphism to handle morphisms of crossed complexes over different groups. For a detailed treatment one needs a discussion of the way that the change of groups functors work with crossed modules or crossed complexes, that is, if we have a morphism of groups $\theta : G \to H$ then we would expect to get functors between Crs_G and Crs_H induced by θ. These do exist and are very nicely behaved, but they will not be discussed here, see [111] for a full treatment in the more general context of profinite groups.

3.4.2 Crossed resolutions and chain resolutions

One of our motivations for introducing crossed complexes was that they enable us to model more of the sort of information encoded in a $K(G,1)$ than does the usual standard algebraic models, e.g. a chain complex such as the bar resolution. In particular, whilst the bar resolution is very good for cohomology with Abelian coefficients for non-Abelian cohomology the crossed version can allow us to push things further, but then comparison on the Abelian theory is very necessary! It is therefore of importance to see how this $K(G,1)$ information that we have encoded changes under the functor $\xi : Crs \to Ch(G{-}Mod)$.

We start with a crossed resolution determined in low dimensions by a presentation $\mathcal{P} = (X : R)$ of a group, G. Thus, in this case, $C_0 = F(X)$ with $\varphi : F(X) \to G$, the 'usual' epimorphism, and $C_1 \to C_0$ is $C \to F(X)$, the free crossed module on $R \to F(X)$. It is not too hard to show that $C_1^{Ab} \cong \mathbb{Z}[G]^{(R)}$, the free $\mathbb{Z}[G]$-module on R. (The proof is left as an exercise.) This maps down onto $N(R)^{Ab}$, the Abelianisation of the normal closure of R in $F(X)$ via a map

$$\partial_* : \mathbb{Z}[G]^{(R)} \to N(R)^{Ab},$$

given by $\partial_*(e_r) = r[N(R), N(R)]$, where e_r is the generator of $\mathbb{Z}[G]$ corresponding to $r \in R$.

There is also a short exact sequence

$$1 \to N(R) \xrightarrow{i} F(X) \xrightarrow{\varphi} G \to 1$$

and hence by Proposition 10, a short exact sequence

$$0 \to N(R)^{Ab} \xrightarrow{\tilde{i}} \mathbb{Z}[G] \otimes_F I(F) \xrightarrow{\tilde{\varphi}} I(G) \to 0$$

(where we have written $F = F(X)$).

By the Corollary to Proposition 8, we have

$$\mathbb{Z}[G] \otimes_F I(F) \cong \mathbb{Z}[G]^{(X)}.$$

The required map $C_1^{Ab} \to D_\varphi$ is the composite

$$\mathbb{Z}[G]^{(R)} \xrightarrow{\partial_*} N(R)^{Ab} \xrightarrow{\tilde{i}} \mathbb{Z}[G]^{(X)}.$$

We have given an explicit description of ∂_* above, so to complete the description of d, it remains to describe $\tilde{i}$, but $\tilde{i}$ satisfies $\tilde{i}\delta = \partial_\varphi i$, where $\delta : N(R) \to N(R)^{Ab}$, so $\tilde{i}(r[N(R), N(R)]) = d_\varphi(r)$. Thus if r is a relator, i.e., if it is in the image of the subgroup generated by the elements of R, then $\partial(e_r)$ can be written as a finite sum of the form $\sum_x a_x e_x$ and the elements $a_x \in \mathbb{Z}[G]$ are the images of the Fox derivatives.

This operator can best be viewed as the Alexander matrix of a presentation of a group, further study of this operator depends on studying transformations between free modules over group rings, and we will not attempt to study those here.

The rest of the crossed resolution does not change and so, on replacing $I(G)$ by $\mathbb{Z}[G] \to \mathbb{Z}$, we obtain a free pseudocompact $\mathbb{Z}[G]$-resolution of the trivial module $\mathbb{Z}$,

$$\ldots \to \mathbb{Z}[G]^{(R)} \xrightarrow{d} \mathbb{Z}[G]^{(X)} \to \mathbb{Z}[G] \to \mathbb{Z}$$

built up from the presentation. This is the complex of chains on the universal cover, $\widetilde{K(G,1)}$, where $K(G, 1)$ is constructed starting from a presentation $\mathcal{P}$.

3.4.3 Standard crossed resolutions and bar resolutions

We next turn to the special case of the standard crossed resolution of G discussed briefly earlier. Of course this is a special case of the previous one, but it pays to examine it in detail.

Clearly in $\xi = \xi(\mathsf{C}G, \varphi)$, we have:
ξ_0 = the free $\mathbb{Z}[G]$-module on the underlying set of G, individual generators being written $[u]$, for $u \in G$;
ξ_1 = the free $\mathbb{Z}[G]$ -module on $G \times G$, generators being written $[u, v]$;
$\xi_n = C_n G$, the free $\mathbb{Z}[G]$ -module on G^{n+1}, etc.

The map $d_2 : \xi_2 \to \xi_1$ induced from δ_2 is given by

$$d_2[u, v, w] = u[v, w] - [u, v] - [uv, w] + [u, vw],$$

and the map $d_1 : \xi_1 \to \xi_0$ by

$$\begin{aligned} d_1([u, v]) &= d_\varphi([uv]^{-1}[u][v]) \\ &= v^{-1}u^{-1}(-[uv] + [u] + u[v]), \end{aligned}$$

a unit times the usual bar resolution formula. Thus, as claimed earlier, the standard crossed resolution is the crossed analogue of the bar resolution.

3.4.4 The intersection $A \cap [C, C]$.

We next turn to a comparison of homological and homotopical syszygies. We have almost all the preliminary work already. The next ingredient is a result that will identify the intersection of the kernel of a crossed module, $A = Ker(C \xrightarrow{\partial} P)$ and the commutator subgroup of C.

The kernel of the homomorphism from A to C^{Ab} is, of course, $A \cap [C, C]$ and this need not be trivial. In fact, Brown and Huebschmann ([32], p.160) note that in examples of type $(G, Aut(G), \partial)$, the kernel of ∂ is, of course, the centre ZG of G and $ZG \cap [G, G]$ can be non-trivial, for instance, if G is dicyclic or dihedral.

We will adopt the same notation as previously with $N = \partial P$ etc.

Proposition 12 *If, in the exact sequence of groups*

$$1 \to A \to C \stackrel{p}{\to} N \to 1,$$

the epimorphism from C to N is split (the splitting need not respect G-action), then $A \cap [C, C]$ is trivial.

Proof: Given a splitting $s : N \to C$, (so ps is the identity on N), then the group C can be written as $A \rtimes s(N)$. The commutators in C, therefore, all lie in $s(N)$ since A is Abelian, but then, of course, $A \cap [C, C]$ cannot contain any non-trivial elements. ■

We used this proposition earlier in the case where N is free. We are thus using the fact that subgroups of free groups are free, in that case. Of course, any epimorphism with codomain a free group is split.

Brown and Huebschmann, [32], p. 168, prove that for an group G with presentation $\mathcal{P}$, the module of identities for $\mathcal{P}$ is naturally isomorphic to the second homology group, $H_2(\tilde{K}(\mathcal{P}))$, of the universal cover of $K(\mathcal{P})$, the 2-complex of the presentation. We can approach this via the algebraic constructions we have.

Given a presentation $\mathcal{P} = \langle X : R\rangle$ of a group G, the algebraic analogue of $K(\mathcal{P})$, we have argued above, is the free crossed module $C(\mathcal{P}) \stackrel{d}{\to} F(X)$ and the chains on the universal cover of $K(\mathcal{P})$ will be given by ξ_G of this, i.e., by the chain complex

$$\mathbb{Z}[G]^{(R)} \stackrel{d}{\to} \mathbb{Z}[G]^{(X)}.$$

In general there will be a short exact sequence

$$0 \to \kappa(\mathcal{P}) \cap [C(\mathcal{P}), C(\mathcal{P})] \to \kappa(\mathcal{P}) \to H_2(\xi(C(\mathcal{P})) \to 0.$$

This short exact sequence yields the Brown-Huebschmann result as $N(R)$ will a free group so the epimorphism onto $N(R)$ splits and we can use the above Proposition 12. We thus get

Proposition 13 *If $\mathcal{P} = \langle X : R\rangle$ is a free presentation of G, then there is an isomorphism*

$$\kappa \xrightarrow{\cong} H_2(\xi(C_{\mathcal{C}}(\mathcal{P})) = Ker(d : \mathbb{Z}[G]^R \to \mathbb{Z}[G]^X).$$

■

Note: Here we are using something that will not be true in all algebraic settings. A subgroup of a free group is always free, but the analogous statement for free algebras of other types is not true.

3.5 Simplicial groups and crossed complexes

3.5.1 From simplicial groups to crossed complexes

Given any simplicial group G, the formula,

$$\mathsf{C}(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

in higher dimensions with, at its 'bottom end', the crossed module,

$$\frac{NG_1}{d_0(NG_2 \cap D_2)} \to NG_0$$

gives a crossed complex with ∂ induced from the boundary in the Moore complex. The detailed proof is too long to indicate here. It just checks the axioms, one by one.

We should have a glance at this formula from various viewpoints, some of which will be revisited later. Once again there is a clear link with the non-uniqueness of fillers for horns in a simplicial group if it is not a group T-complex. We have all those $(NG_n \cap D_n)$ terms involved!

Suppose that we had our simplicial group G and wanted to construct a quotient of it that was a group T-complex. We could do this in a silly way since the trivial simplicial group is clearly a group T-complex, but let us keep the quotient as large as possible. This problem is related to the question of whether the category of group T-complexes forms a reflexive subcategory of *Simp.Grps*. The condition $NG \cap D = 1$ looks like some sort of 'equational specification'. Our question can thus really be posed as follows: Suppose we have a simplicial group morphism $f : G \to H$ and H is a group T-complex. Remember that in group T-complexes, as against the non-algebraic ones, the thin structure is not an added bit of structure. The thin elements are determined by the degeneracies, so whether or not H is or is not a group T-complex is somehow its own affair, and nothing to do with any external factors! Does f factor universally through some 'group T-complexification' of G? Something like

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "proj"'] & & H \\ & G/T(G) \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

with $G/T(G)$ a group T-complex and $\hat{f}$ uniquely determined by the diagram.

One sensible way to look at such a question is to assume, provisionally, that such a factorisation exists and to see what $T(G)$ would have to be. In general, if $f : G \to H$ is any simplicial group morphism (with no restriction on H for the moment), then with a hopefully obvious notation,

$$f_n(NG_n \cap D(G)_n) \subseteq NH_n \cap D(H)_n,$$

since f sends degenerate elements to degenerate elements and preserves products! Back in our situation in which H is a group T-complex, then $f_n(NG_n \cap D(G)_n) = 1$, for the simple reason that the right hand side of that displayed formula is trivial by assumption. We thus have that if some such $T(G)$ exists, then we must have $NG_n \cap D(G)_n \subseteq T(G)_n$ and our first attempt might be to look at the possibility that they should be equal. This is wrong and for fairly trivial reasons. The subgroup $T(G)_n$ of G_n has to be normal if we are to form the quotient by it, and there is no reason why $NG_n \cap D(G)_n$ should be a normal subgroup in general.

We might then be tempted to take the normal subgroup generated by $NG_n \cap D(G)_n$, but that is 'defeatist' in this situation. We might hope to do detailed calculations with the subgroup and if it is specified as a normal closure, we will lose some of our ability to do that, at least without considerable more effort. (Let's be lazy and see if we can get around that difficulty.) If we look again, we find another thing that 'goes wrong' with any attempt to use $NG_n \cap D(G)_n$ as it is. This subgroup would be within NG_n, of course, and we want to induce a map from the Moore complex of G to that of $G/T(G)$. For that to work, we would need not only $NG_n \cap D(G)_n \subseteq T(G)_n$, but the image of $NG_n \cap D(G)_n$ under d_0 to be in $T(G)_{n-1}$. Going up a dimension, we thus need not only $NG_n \cap D(G)_n$, but $d_0(NG_{n+1} \cap D(G)_{n+1}) \subseteq T(G)_n$. We thus need the product subgroup $(NG_n \cap D(G)_n)d_0(NG_{n+1} \cap D(G)_{n+1})$ to be in $T(G)_n$. This looks a bit complicated. Do we need to go any further up the Moore complex? No, because d_0d_0 is trivial. We might thus try

$$T(G)_n = (NG_n \cap D(G)_n)d_0(NG_{n+1} \cap D(G)_{n+1})$$

You might now think that this is a bit silly because we would still need this product subgroup to be normal in order to form the quotient ... , but it is! The lack of normality of our earlier attempt is absorbed by the image of the next level up. (That is pretty!)

Of course, there are very good reasons why this works. These involve what are sometimes called *Peiffer pairings*. We will see some of these later.

As a consequence of the above discussion, we more or less have:

Proposition 14 *If G is a group T-complex, then NG is a crossed complex.* ∎

We certainly have a sketch of

Proposition 15 *The full subcategory of Simp.Grps determined by the group T-complexes is a reflective subcategory.* ∎

Of course, the details of the proofs of both of these are left for you to write down. Nearly all of the reasoning for the second result is there for you, but some of the detailed calculations for the first are quite tricky.

The close link between group T-complexes and crossed complexes is evident from these results. You might guess that they form equivalent categories. They do. We will look at the way back from crossed complexes (of groups) to simplicial groups later on, but we need to get back to cohomology.

3.5.2 Simplicial resolutions, a bit of background

We need some such means of going from simplicial groups to crossed complexes so because we can also use simplicial resolutions to 'resolve' a group (and in many other situations). We first sketch in some historical background.

In the 1960s, the connection between simplicial groups and cohomology was examined in detail. The basic idea was that given the adjoint "free-forget" pair of functors between *Groups* and *Sets*, one could generate a free resolution of a group, G, using the resulting comonad (or cotriple) (cf. MacLane, [92]). This resolution was not, however, by a chain complex but by a free simplicial group, F, say. It was then shown (Barr and Beck, [11]) that given any G-module, M, and working in the category of groups over G, one could form the cosimplicial G-module, $Hom_{Gps/G}(F, M)$, and hence, by a dual form of the Dold-Kan theorem, a cochain complex $C(G, M)$, whose homotopy type, and hence whose homology, was independent of the choice of F. This homology was the usual

Eilenberg-MacLane cohomology of G with coefficients in M, but with a shift in dimension (cf. Barr and Beck, [11]).

Other theories of cohomology were developed at about the same time by Grothendieck and Verdier, [5], André, [3, 4], and Quillen, [113, 114]. The first of these was designed for use with "sites", that is, categories together with a Grothendieck topology.

André and Quillen developed, independently, a method of defining cohomology using simplicial resolutions. Their work is best known in commutative algebra, but their methods work in greater generality. Unlike the theory of Barr and Beck (monadic cohomology), they only assume there is enough structure to construct free resolutions; a (co)monad is just one way of doing this. In particular, André, [3, 4], describes a step-by-step, almost combinatorial, process for constructing such resolutions. This ties in well with our earlier comments about using a presentation of a group to construct a crossed resolution and the important link with syzygies. André's method is the simplicial analogue of this.

We will assume for the moment that we have a simplicial resolution, F, of our group, G. Both André and Quillen then consider applying a derived module construction dimensionwise to F, obtaining a simplicial G-module. They then use "Dold-Kan" to give a chain complex of G-modules, which they call the "cotangent complex", denoted L_G or $LAb(G)$, of G (at least in the case of commutative algebras). The homotopy type of $LAb(G)$ does not depend on the choice of resolution and so is a useful invariant of G. We will need to look at this construction in more detail, but will consider a slightly more general situation to start with.

3.5.3 Free simplicial resolutions

Standard theory (cf. Duskin, [56]) shows that if F and F' are free simplicial resolutions of groups, G and H, say, and $f : G \to H$ is a morphism, then f can be lifted to $f' : F \to F'$. The method is the simplicial analogue of lifting a homomorphism of modules to a map of resolutions of those modules, which you should look at first as it is technically simpler. Any two such lifts are homotopic (by a simplicial homotopy).

Of course, f will also lift to a morphism of crossed complexes, $f : \mathsf{C}(F) \to \mathsf{C}(F')$, and any two such lifts will be *homotopic* as crossed complex morphisms. Thus whatever simplicial lift, $f' : F \to F'$, we choose, $\mathsf{C}(f')$ will be a lift in the "crossed" case, and although we do not know at this stage of our discussion of the theory if a homotopy between two simplicial lifts is transferred to a homotopy between the images under C, this does not matter as the *relation* of homotopy is preserved at least in this case of resolutions.

Any group has a free simplicial resolution. There is the obvious adjoint pair of functors

$$\begin{aligned} U &: Groups \to Sets \\ F &: Sets \to Groups \end{aligned}$$

Writing $\eta : Id \to UF$ and $\varepsilon : FU \to Id$ for the unit and counit of this adjunction (cf. MacLane, [92, 93]), we have a comonad (or cotriple) on *Groups*, the free group comonad, $(FU, \varepsilon, F\eta U)$. We write $L = FU$, $\delta = F\eta U$, so that

$$\varepsilon : L \to I$$

is the counit of the comonad whilst

$$\delta : L \to L^2$$

is the comultiplication. (For the reader who has not met monads or comonads before, (L, η, δ) behaves as if it was a monoid in the dual of the category of "endofunctors" on *Groups*, see MacLane, [93] Chapter VI. We will explore them briefly in section **??**, starting on page **??**.)

Now suppose G is a group and set $F(G)_i = L^{i+1}(G)$, so that $F(G)_0$ is the free group on the underlying set of G and so on. The counit (which is just the augmentation morphism from $FU(G)$ to G) gives, in each dimension, face morphisms

$$d_i = L^{n-i}\varepsilon L^i(G) : L^{n+1}(G) \to L^n(G),$$

for $i = 0, \dots, n$, whilst the comultiplication gives degeneracies

$$s_i : L^n(G) \to L^{n+1}(G)$$

$$s_i = L^{n-1-i}\delta L^i,$$

for $i = 0, \dots, n-1$, satisfying the simplicial identities.

Remark: Here we follow the conventions used by Duskin, in his Memoir, [56]. Later we will also need to look at similar resolutions where the labelling of the faces and degeneracies are reversed.

This simplicial group, $F(G)$, satisfies $\pi_0(F(G)) \cong G$ (the isomorphism being induced by $\varepsilon(G) : F_0(G) \to G$) and $\pi_n(F(G))$ is trivial if $n \geq 1$. The reason for this is simple. If we apply U once more to $F(G)$, we get a simplicial set and the unit of the adjunction

$$\eta : 1 \to UF$$

allows one to define for each n

$$\eta U(FU)^n : UL^n \to UL^{n+1},$$

which gives a natural contraction of the augmented simplicial set, $UF(G) \to U(G)$, (cf. Duskin, [56]). We will look at this in detail in our later treatment of augmentations, etc. For the moment, it suffices to accept the fact that we do get a resolution, as we do not need to know the details of why this construction works, at least not yet.

If we denote the constant simplicial group on G by $K(G, 0)$, the augmentation defines a simplical homomorphism

$$\overline{\varepsilon} : F(G) \to K(G, 0)$$

satisfying $U\overline{\varepsilon}.inc = Id$, where $inc : UK(G, 0) \to UF(G)$ is the 'inclusion' of simplicial sets given by η, and then these extra maps, $(UF)^n\eta U$, in fact, give a homotopy between $inc.U\overline{\varepsilon}$ and the identity map on $UF(G)$, i.e., $\overline{\varepsilon}$ is a weak homotopy equivalence of simplicial groups. Thus $F(G)$ is a free simplicial resolution of G. It is called the *comonadic free simplicial resolution* of G.

This simplicial resolution has the advantage of being functorial, but the disadvantage of being very big. We turn next to a 'step-by-step' method of constructing a simplicial resolution using ideas pioneered by André, [4], although most of his work was directed more towards commutative algebras, cf. [3].

3.5.4 Step-by-Step Constructions

This section is a brief résumé of how to construct simplicial resolutions by hand rather than functorially. This allows a better interpretation of the generators in each level of the resolution. These are the simplicial analogues of higher syzygies. The work depends heavily on a variety of sources, mainly [3], [86] and [101]. André only treats commutative algebras in detail, but Keune [86] does discuss the general case quite clearly. The treatment here is adapted from the paper by Mutlu and Porter, [102].

Recall of notation: We first recall some notation and terminology, which will be used in the construction of a simplicial resolution. Let $[n]$ be the ordered set, $[n] = \{0 < 1 < \cdots < n\}$. Define the following maps: the injective monotone map $\delta_i^n : [n-1] \to [n]$ is given by

$$\delta_i^n(k) = \begin{cases} k & \text{if} \quad k < i, \\ k+1 & \text{if} \quad k \geq i, \end{cases}$$

for $0 \leq i \leq n \neq 0$. The increasing surjective monotone map $\alpha_i^n : [n+1] \to [n]$ is given by

$$\alpha_i^n(k) = \begin{cases} k & \text{if } k \leq i, \\ k-1 & \text{if } k > i, \end{cases}$$

for $0 \leq i \leq n$. We denote by $\{m, n\}$ the set of increasing surjective maps $[m] \to [n]$.

3.5.5 Killing Elements in Homotopy Groups

Let G be a simplicial group and let $k \geq 1$ be fixed. Suppose we are given a set, Ω, of elements: $\Omega = \{x_\lambda : \lambda \in \Lambda\}$, $x_\lambda \in \pi_{k-1}(\mathsf{G})$, then we can choose a corresponding set of elements $\theta_\lambda \in NG_{k-1}$ so that $x_\lambda = \theta_\lambda\, \partial_k(NG_k)$. (If $k = 1$, then as $NG_0 = G_0$, the condition that $\theta_\lambda \in NG_0$ is immediate.) We want to 'kill' the elements in Ω.

We form a new simplicial group F_n where

1) F_n is the free G_n-group, (i.e., group with G_n-action)

$$F_n = \coprod_{\lambda,t} G_n\{y_{\lambda,t}\} \text{ with } \lambda \in \Lambda \text{ and } \; t \in \{n, k\},$$

where $G_n\{y\} = G_n * < y >$, the co-product of G_n and a free group generated by y.

2) For $0 \leq i \leq n$, the group homomorphism $s_i^n : F_n \to F_{n+1}$ is obtained from the homomorphism $s_i^n : G_n \to G_{n+1}$ with the relations

$$s_i^n(y_{\lambda,t}) = y_{\lambda,u} \quad \text{with} \quad u = t\alpha_i^n, \quad t : [n] \to [k].$$

3) For $0 \leq i \leq n \neq 0$, the group homomorphism $d_i^n : F_n \to F_{n-1}$ is obtained from $d_i^n : G_n \to G_{n-1}$ with the relations

$$d_i^n(y_{\lambda,t}) = \begin{cases} y_{\lambda,u} & \text{if the map} \quad u = t\delta_i^n \quad \text{is surjective,} \\ t'(\theta_\lambda) & \text{if} \quad u = \delta_k^k t', \\ 1 & \text{if} \quad u = \delta_j^k t' \quad \text{with} \;\; j \neq k, \end{cases}$$

by extending multiplicatively.

We sometimes denote the F, so constructed by $\mathsf{G}(\Omega)$.

Remark: In a 'step-by-step' construction of a simplicial resolution, (see below), there will thus be the following properties: i) $F_n = G_n$ for $n < k$, ii) F_k = a free G_k-group over a set of non-degenerate indeterminates, all of whose faces are the identity except the k^{th}, and iii) F_n is a free G_n-group on some degenerate elements for $n > k$.

We have immediately the following result, as expected.

Proposition 16 *The inclusion of simplicial groups $G \hookrightarrow F$, where $F = G(\Omega)$, induces a homomorphism*

$$\pi_n(G) \longrightarrow \pi_n(F)$$

for each n, which for $n < k-1$ is an isomorphism,

$$\pi_n(G) \cong \pi_n(F)$$

and for $n = k-1$, is an epimorphism with kernel generated by elements of the form $\bar{\theta}_\lambda = \theta_\lambda \partial_k NG_k$, where $\Omega = \{x_\lambda : \lambda \in \Lambda\}$. ■

3.5.6 Constructing Simplicial Resolutions

The following result is essentially due to André, [3].

Theorem 4 *If G is a group, then it has a free simplicial resolution $\mathbb{F}$.*

Proof: The repetition of the above construction will give us the simplicial resolution of a group. Although 'well known', we sketch the construction so as to establish some notation and terminology.

Let G be a group. The zero step of the construction consists of a choice of a free group F and a surjection $g : F \to G$ which gives an isomorphism $F/Ker\, g \cong G$ as groups. Then we form the constant simplicial group, $F^{(0)}$, for which in every degree n, $F_n = F$ and $d_i^n = \mathrm{id} = s_j^n$ for all i, j. Thus $F^{(0)} = K(F, 0)$ and $\pi_0(F^{(0)}) = F$. Now choose a set, Ω^0, of normal generators of the closed normal subgroup $N = Ker\,(F \xrightarrow{g} G)$, and obtain the simplicial group in which $F_1^{(1)} = F(\Omega^0)$ and for $n > 1$, $F_n^{(1)}$ is a free F_n-group over the degenerate elements as above. This simplicial group will be denoted by $F^{(1)}$ and will be called the *1-skeleton of a simplicial resolution of the group G*.

The subsequent steps depend on the choice of sets, Ω^0, $\Omega^1, \Omega^2, \ldots, \Omega^k, \ldots$. Let $F^{(k)}$ be the simplicial group constructed after k steps, that is, *the k-skeleton of the resolution*. The set Ω^k is formed by elements a of $F_k^{(k)}$ with $d_i^k(a) = 1$ for $0 \le i \le k$ and whose images $\bar{a}$ in $\pi_k(F^{(k)})$ generate that module over $F_k^{(k)}$ and $F^{(k+1)}$.

Finally we have inclusions of simplicial groups

$$F^{(0)} \subseteq F^{(1)} \subseteq \cdots \subseteq F^{(k-1)} \subseteq F^{(k)} \subseteq \cdots$$

and in passing to the inductive limit (colimit), we obtain an acyclic free simplicial group F with $F_n = F_n^{(k)}$ if $n \le k$. This F, or, more exactly, (F, g), is thus a simplicial resolution of the group G.

The proof of theorem is completed. ■

Remark: A variant of the 'step-by-step' construction gives: *if G is a simplicial group, then there exists a free simplicial group F and a continuous epimorphism $F \longrightarrow G$ which induces isomorphisms on all homotopy groups.* The details are omitted as they should be reasonably clear.

The key observation, which follows from the universal property of the construction, is a freeness statement:

Proposition 17 *Let $F^{(k)}$ be a k-skeleton of a simplicial resolution of G and $(\Omega^k, g^{(k)})$ k-dimension construction data for $F^{(k+1)}$. Suppose given a simplicial group morphism $\Theta : F^{(k)} \longrightarrow G$ such that $\Theta_*(g^{(k)}) = 0$, then Θ extends over $F^{(k+1)}$.*

This freeness statement does not contain a uniqueness clause. That can be achieved by choosing a lift for $\Theta_k g^{(k)}$ to NG_{k+1}, a lift that must exist since $\Theta_*(\pi_k(F^{(k)}))$ is trivial.

When handling combinatorially defined resolutions, rather than functorially defined ones, this proposition is as often as close to 'left adjointness' as is possible without entering the realm of homotopical algebra to an extent greater than is desirable for us here.

We have not talked here about the homotopy of simplicial group morphisms, and so will not discuss homotopy invariance of this construction for which one adapts the description given by André, [3], or Keune, [86]. Of course, the resolution one builds by any means would be homotopicallly equivalent to any other so, for cohomological purposes, it makes no difference how the resolution is built.

Of course, from any simplicial resolution F of G, you can get an augmented crossed complex $C(F)$ over G using the formula given earlier and this is a crossed resolution.

3.6 Cohomology and crossed extensions

3.6.1 Cochains

Consider a G-module, M, and a non-negative integer n. We can form the chain complex, $K(M, n)$, having M in dimension n and zeroes elsewhere. We can also form a crossed complex, $\mathsf{K}(M, n)$, that plays the role of the n^{th} Eilenberg-MacLane space of M in this setting. We may call it the n^{th} *Eilenberg-MacLane crossed complex* of M:

If $n = 0$, $\mathsf{K}(M, n)_0 = M \rtimes G$, $\mathsf{K}(M, n)_i = 0$, $i > 0$.

If $n \geq 1$, $\mathsf{K}(M, n)_0 = G$, $\mathsf{K}(M, n)_n = M$, $\mathsf{K}(M, n)_i = 0$, $i \neq 0$ or n.

One way to view cochains is as chain complex morphisms. Thus on looking at $Ch(\mathsf{B}G, K(M, n))$, one finds exactly $Z^{n+1}(G, M)$, the $(n + 1)$-cocycles of the cochain complex $C(G, M)$. We can also view $Z^{n+1}(G, M)$ as $Crs_G(\mathsf{C}G, \mathsf{K}(M, n))$.

In the category of chain complexes, one has that a homotopy from $\mathsf{B}G$ to $K(M, n)$ between 0 and f, say, is merely a coboundary, so that $H^{n+1}(G, M) \cong [\mathsf{B}G, K(M, n)]$, adopting the usual homotopical notation for the group of homotopy classes of maps from the bar resolution $\mathsf{B}G$ to $K(M, n)$. This description has its analogue in the crossed complex case as we shall see.

3.6.2 Homotopies

Let C, C' be two crossed complexes with Q and Q' respectively as the cokernels of their bottom morphism. Suppose λ, $\mu : \mathsf{C} \to \mathsf{C}'$ are two morphisms inducing the same map $\varphi : Q \to Q'$.

A homotopy from λ to μ is a family, $h = \{h_k : k \geq 1\}$, of maps $h_k : C_k \to C'_{k+1}$ satisfying the following conditions:

H1) $h_0 : C_1 \to C'_2$ is a derivation along μ_0 (i.e. for $x, y \in C_0$,

$$h_0(xy) = h_0(x)({}^{\mu_0}h_0(y)),)$$

such that

$$\delta_1 h_0(x) = \lambda_0(x)\mu_0(x)^{-1}, \quad x \in C_0.$$

H2) $h_1 : C_1 \to C_2'$ is a C_0-homomorphism with C_0 acting on C_2' via λ_0 (or via μ_0, it makes no difference) such that

$$\delta_2 h_1(x) = \mu_1(x)^{-1}(h_0\delta_1(x)^{-1}\lambda_1(x)) \text{ for } x \in C_1.$$

H3) for $k \geq 2$, h_k is a Q-homomorphism (with Q acting on the C_k' via the induced map $\varphi : Q \to Q'$) such that

$$\delta_{k+1}h_k + h_{k-1}\delta_k = \lambda_k - \mu_k.$$

We note that the condition that λ and μ induce the same map, $\varphi : Q \to Q'$, is, in fact, superfluous as this is implied by $H1$.

The properties of homotopies and the relation of homotopy are as one would expect. One finds $H^{n+1}(G, M) \cong [\mathsf{C}G, \mathsf{K}(M, n)]$. Given that in higher dimensions, this is the same set exactly as $[\mathsf{B}G, K(M, n)]$ means that there is not much to check and so the proof has been omitted.

3.6.3 Huebschmann's description of cohomology classes

The transition from this position to obtaining Huebschmann's descriptions of cohomology classes, [77], is now more or less formal. We will, therefore, only sketch the main points.

If G is a group, M is a G-module and $n \geq 1$, a *crossed n-fold extension* is an exact augmented crossed complex,

$$0 \to M \to C_n \to \ldots \to C_2 \to C_1 \to G \to 1.$$

The notion of similarity of such extensions is analogous to that of n-fold extensions in the Abelian Yoneda theory, (cf. MacLane, [92]), as is the definition of a Baer sum. We leave the details to you. This yields an Abelian group, $Opext^n(G, M)$, of similarity classes of crossed n-fold extensions of G by M.

Given a cohomology class in $H^{n+1}(G, M)$ realisable as a homotopy class of maps, $f : \mathsf{C}G \to \mathsf{K}(M, n)$, one uses f to form an induced crossed complex, much as in the Abelian Yoneda theory:

$$\begin{tikzcd}
 & J_n(G) \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\text{pushout}"] & C_n \arrow[r] \arrow[d] & \cdots \arrow[r] & C_1 \arrow[r] \arrow[d] & G \arrow[d, equal] \\
0 \arrow[r] & M \arrow[r] & M_n \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r] & G
\end{tikzcd}$$

where $J_n(G)$ is $Ker(C_nG \to C_{n-1}G)$. (Thus J_nG is also $Im(C_{n+1}G \to C_nG)$ and as the map f satisfies $f\delta = 0$, it is zero on the subgroup $\delta(C_{n+2}G)$ (i.e. is constant on the cosets) and hence passes to $Im(C_{n+1}G \to C_nG)$ in a well defined way.) Arguments using lifting of maps and homotopies show that the assignment of this element of $Opext^n(G, M)$ to $cls(f) \in H^{n+1}(G, M)$ establishes an isomorphism between these groups.

3.6.4 Abstract Kernels.

The importance of having such a description of classes in $H^n(G, M)$ probably resides in low dimensions. To describe classes in $H^3(G, M)$, one has, as before, crossed 2-fold extensions

$$0 \to M \to C_2 \xrightarrow{\partial} C_1 \to G \to 1,$$

where ∂ is a crossed module. One has for any group G, a crossed 2-fold extension

$$0 \to Z(G) \to G \stackrel{\partial_G}{\to} Aut(G) \to Out(G) \to 1$$

where ∂_G sends $g \in G$ to the corresponding inner automorphism of G. An *abstract kernel* (in the sense of Eilenberg-MacLane, [61]) is a homomorphism $\psi : Q \to Out(G)$ and hence provides, by pulling back, a 2-fold extension of Q by the centre, $Z(G)$, of G.

3.7 2-types and cohomology

In classifying homotopy types and in obstruction theory, one frequently has invariants that are elements in cohomology groups of the form $H^m(X, \pi)$, where typically π is the n^{th} homotopy group of some space. When dealing with homotopy types, π will be a group, usually Abelian with a π_1-action, i.e., we are exactly in the situation described earlier, except that X is a homotopy type not a group. Of course, provided that X is connected, we can replace X by a simplicial group, bringing us even nearer to the situation of this section. We shall work within the category of simplicial groups.

3.7.1 2-types

A morphism

$$f : G \to H$$

of simplicial groups is called a *2-equivalence* if it induces isomorphisms

$$\pi_0(f) : \pi_0(G) \to \pi_0(H,)$$

and

$$\pi_1(f) : \pi_1(G) \to \pi_1(H).$$

We can form a quotient category, $Ho_2(Simp.Grps)$, of *Simp.Grps* by formally inverting the 2-equivalences, then we say two simplicial groups, G and H, have the same *2-type*, (or, more exactly, *homotopy 2-type*), if they are isomorphic in $Ho_2(Simp.Grps)$.

This is, of course, just a special case of the general notion of n-type in which "n-equivalences" are inverted, thus forming the quotient category $Ho_n(Simp.Grps)$.

We recall the following from earlier:

Definition: An *n-equivalence* is a morphism, f, of simplicial groups (or groupoids) inducing isomorphisms, $\pi_i(f)$, for $i = 0, 1, \ldots, n-1$.

Definition: Two simplicial groups, G and H, have the same *n-type* (or, more exactly, *homotopy n-type* if they are isomorphic in $Ho_n(Simp.Grps)$.

Sometimes it is convenient to say that a simplicial group, G, *is* an n-type. This is taken to mean that it represents an n-equivalence class *and* has zero homotopy groups above dimension $n-1$.

3.7.2 Example: 1-types

Before examining 2-types in detail, it will pay to think about 1-types. A morphism f as above is a 1-equivalence if it induces an isomorphism on π_0, i.e., $\pi_0(f)$ is an isomorphism. Given any group G, there is a simplicial group, $K(G,0)$ consisting of G in each dimension with face and degeneracy maps all being identities. Given a simplicial group, H, having $G \cong \pi_0(H)$, the natural quotient map

$$H_0 \to \pi_0(H) \cong G,$$

extends to a natural 1-equivalence between H and $K(\pi_0(H),0)$.

It is fairly routine to check that

$$\pi_0 : Simp.Grps \to Grps$$

has $K(-,0)$ as an adjoint and that, as the unit is a natural 1-equivalence, and the counit an isomorphism, this adjoint pair induces an equivalence between the category $Ho_1(Simp.Grps)$ of 1-types and the category, *Grps*, of groups. In other words,

groups are algebraic models for 1-types.

3.7.3 Algebraic models for n-types?

So much for 1-types. Can one provide algebraic models for 2-types or, in general, n-types? We touched on this earlier. The criteria that any such "models" might satisfy are debatable. Perhaps ideally, or even unrealistically, there should be an isomorphism class of algebraic "gadgets" for each 2-type. An alternative weaker solution is to ask that a notion of equivalence between the models is possible, and that only equivalence classes, not isomorphism classes, correspond to 2-types, but, in addition, the notion of equivalence is algebraically defined. It is this weaker possibility that corresponds to our aim here.

3.7.4 Algebraic models for 2-types.

If G is a simplicial group, then we can form a crossed module

$$\partial : \frac{NG_1}{d_0(NG_2)} \to G_0,$$

where the action of G_0 is via the degeneracy, $s_0 : G_0 \to G_1$, and ∂ is induced by d_0. (As before we will denote this crossed module by $M(G,1)$.) The kernel of ∂ is

$$\frac{Ker\, d_0 \cap Ker\, d_1}{d_0(NG_2)} \cong \pi_1(G),$$

whilst its cokernel is

$$\frac{G_0}{d_0(NG_1)} \cong \pi_0(G),$$

and so we have a crossed 2-fold extension

$$0 \to \pi_1(G) \to \frac{NG_1}{d_0(NG_2)} \to G_0 \to \pi_0(G) \to 1$$

and hence a cohomology class $k(G) \in H^3(\pi_0(G), \pi_1(G))$.

Suppose now that $f : G \to H$ is a morphism of simplicial groups, then one obtains a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(G) \arrow[r] \arrow[d, "\pi_1(f)"'] & \frac{NG_1}{d_0(NG_2)} \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d, "f_0"] & \pi_0(G) \arrow[r] \arrow[d, "\pi_0(f)"] & 1 \\ 0 \arrow[r] & \pi_1(H) \arrow[r] & \frac{NH_1}{d_0(NH_2)} \arrow[r] & H_0 \arrow[r] & \pi_0(H) \arrow[r] & 1 \end{tikzcd}$$

If, therefore, f is a 2-equivalence, $\pi_0(f)$ and $\pi_1(f)$ will be isomorphisms and the diagram shows that, modulo these isomorphisms, $k(G)$ and $k(H)$ are the same cohomology class, i.e. the 2-type of G determines π_0, π_1 and this cohomology class, k in $H^3(\pi_0, \pi_1)$.

Conversely, suppose we are given a group π, a π-module, M, and a cohomology class $k \in H^3(\pi, M)$, then we can realise k by a 2-fold extension

$$0 \to M \to C \stackrel{\partial}{\to} G \to \pi \to 1$$

as above.

The crossed module, $\mathsf{C} = (C, G, \partial)$, determines a simplicial group $K(\mathsf{C})$ as follows:

Suppose $\mathsf{C} = (C, P, \partial)$ is any crossed module, we construct a simplicial group, $K(\mathsf{C})$, by

$$K(\mathsf{C})_0 = P, \qquad K(\mathsf{C})_1 = C \rtimes P,$$

$$s_0(p) = (1, p),\ d_0^1(c, p) = \partial c.p,\ d_1^1(c, p) = p.$$

Assuming $K(\mathsf{C})_n$ is defined and that it acts on C via the unique composed face map to $K(\mathsf{C})_0 = P$ followed by the given action of P on C, we set

$$\begin{aligned} K(\mathsf{C})_{n+1} &= C \rtimes K(\mathsf{C})_n; \\ d_0^{n+1}(c_{n+1}, \ldots, c_1, p) &= (c_{n+1}, ..., c_2, \partial c_1.p); \\ d_i^{n+1}(c_{n+1}, \ldots, c_{i+1}, c_i, \ldots, c_1, p) &= (c_{n+1}, \ldots, c_{i+1}c_i, \ldots c_1, p) \\ &\qquad\qquad \text{for } 0 < i < n+1; \\ d_{n+1}^{n+1}(c_{n+1}, \ldots, c_1, p) &= (c_n, \ldots, c_1, p); \\ s_i^n(c_n, \ldots, c_1, p) &= (c_n, \ldots, 1, \ldots, c_1, p), \end{aligned}$$

where the 1 is placed in the i^{th} position.

Clearly $Ker\, d_1^1 = \{(c, p) : p = 1\} \cong C$, whilst $Ker\, d_1^2 \cap Ker\, d_2^2 = \{(c_2, c_1, p) : (c_1, p) = (1, 1) \text{ and } (c_2c_1, p) = (1, 1)\} \cong \{1\}$, hence the "top term" of $M(K(\mathsf{C}), 1)$ is isomorphic to C itself, whilst $K(\mathsf{C})_0$ is P itself. The boundary map ∂ in this interpretation is the original ∂, since it maps $(c, 1)$ to $d_0(c)$, i.e., we have

Lemma 14 *There is a natural isomorphism*

$$\mathsf{C} \cong M(K(\mathsf{C}), 1).$$

■

This construction is the internal nerve of the corresponding internal category in *Grps*, as we noted earlier. All the ideas that go into defining the nerve of a category adapt to handling internal categories, and they produce simplicial objects in the corresponding ambient category. As we have a simplicial group $K(\mathsf{C})$, we might check if it is a group T-complex, but this is more or less immediate as $NK(\mathsf{C})_n = 1$ for $n \geq 2$, whilst $NK(\mathsf{C})_1$ is $\{(c,p) : p = 1\}$ and $s_0(K(\mathsf{C})_0 = \{(c,p) : c = 1\}$.

Suppose now that we had chosen an equivalent 2-fold extension

$$0 \to M \to C' \stackrel{d'}{\to} G' \to \pi \to 1$$

The equivalence guarantees that there is a zig-zag of maps of 2-fold extensions joining it to that considered earlier. We need only look at the case of a direct basic equivalence:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & C \arrow[r, "\partial"] \arrow[d] & G \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d, "="] & 1 \\
0 \arrow[r] & M \arrow[r] & C' \arrow[r, "\partial'"] & G' \arrow[r] & \pi \arrow[r] & 1
\end{tikzcd}$$

giving a map of crossed modules, $\varphi : \mathsf{C} \to \mathsf{C}'$, where $\mathsf{C}' = (C', G', \partial')$. This induces a morphism of simplicial groups,

$$K(\varphi) : K(\mathsf{C}) \to K(\mathsf{C}'),$$

that is, of course, a 2-equivalence. If there is a longer zig-zag between C and C', then the intermediate crossed modules give intermediate simplicial groups and a zig-zag of 2-equivalences so that $K(\mathsf{C})$ and $K(\mathsf{C}')$ are isomorphic in $Ho_2(Simp.Grps)$, i.e. they have the same 2-type. This argument can, of course, be reversed.

If G and H have the same 2-type, they are isomorphic within the category $Ho_2(Simp.Grps)$, so they are linked in *Simp.Grps* by a zig-zag of 2-equivalences, hence the corresponding cohomology classes in $H^3(\pi_0(G), \pi_1(G))$ are the same up to identification of $H^3(\pi_0(G), \pi_1(G))$ and $H^3(\pi_0(H), \pi_1(H))$. This proves the simplicial group analogue of the result of MacLane and Whitehead, [95], that we mentioned earlier, giving an algebraic model for 2-types of connected CW-complexes.

Theorem 5 *(MacLane and Whitehead, [95]) 2-types are classified by a group π_0, a π_0-module, π_1 and a class in $H^3(\pi_0, \pi_1)$.* ■

We have handled this in such a way so as to derive an equivalence of categories:

Proposition 18 *There is an equivalence of categories,*

$$Ho_2(Simp.Grps) \cong Ho(CMod),$$

where $Ho(CMod)$ is formed from $CMod$ by formally inverting those maps of crossed modules that induce isomorphisms on both the kernels and the cokernels. ■

3.8 Re-examining group cohomology with Abelian coefficients

3.8.1 Interpreting group cohomology

We have had

- A definition of group cohomology via the bar resolution: for a group G and a G-module, M:

$$H^n(G,M) = H^n(C(G,M))$$

together with an identification of $C(G,M)$ with maps from the classifying space / nerve, BG, of G to M, up to shifts in dimension;

- Interpretations

$$\begin{array}{rcl} H^0(G,M) & \cong & M^G, \text{ the module of invariants} \\ H^1(G,M) & \cong & Der(G,M)/Pder(G,M) \\ & - & \text{by inspection, where } Pder(G,M) \text{ is the submodule of} \\ & & \qquad \text{principal derivations;} \\ H^2(G,M) & \cong & Opext(G,M), \text{ i.e. classes of extensions} \\ & & \qquad 0 \to M \to H \to G \to 1 \end{array}$$

and we also have

$$\begin{array}{rcl} H^n(G,M) & \cong & Opext^n(G,M), n \geq 2, \text{ via crossed resolutions} \\ & \cong & [\mathsf{C}(G), \mathsf{K}(M,n)] \end{array}$$

Another interpretation, which will be looked at shortly is as $Ext^n(\mathbb{Z}, M)$, where $\mathbb{Z}$ is given the trivial G-module structure. This leads to

$$H^n(G,M) \cong Ext^{n-1}(I(G), M),$$

via the long exact sequence coming from

$$0 \to I(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

3.8.2 The *Ext* long exact sequences

There are several different ways of examining the long exact sequence that we need. We will use fairly elementary methods rather than more 'homologically intensive' one. These latter ones are very elegant and very powerful, but do need a certain amount of development before being used. The more elementary ones have, though, a hidden advantage. The intuitions that they exploit are often related to ones that extend, at least partially, to the non-Abelian case and also to the geometric situations that will be studied later in the notes.

The idea is to explore what happens to an exact sequence of modules

$$\mathcal{E}: \quad 0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

over some given ring (we need it for G-modules so there the ring is $\mathbb{Z}[G]$, the group ring of G), when we apply the functor $Hom(-, M)$ for M another module. Of course one gets a sequence

$$Hom(\mathcal{E}, M): 0 \to Hom(C,M) \overset{\beta^*}{\to} Hom(B,M) \overset{\alpha^*}{\to} Hom(A,M)$$

and it is easy to check that this is exact, but there is no reason why α^* should be onto since a morphism $f : A \to M$ may or may not extend to some g defined over the bigger module B. For instance, if $M = A$, and f is the identity morphism, then f extends if and only if the sequence splits (so $B \cong A \oplus C$). We examine this more closely.

We have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

and can form a new diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "\overline{f}"] & C \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & M \arrow[r, "\overline{\alpha}"] & N \arrow[r, "\overline{\beta}"] & C \arrow[r] & 0 \end{tikzcd}$$

where the left hand square is a pushout. You should check that you see why there is an induced morphism $\overline{\beta} : N \to C$ 'emphusing the universal property of pushouts. (This is important as sometimes one wants this sort of construction, or argument, for sheaves of modules and there working with elements causes some slight difficulties.) The existence of this map is guaranteed by the universal property and does not depend on a particular construction of N. Of course this means that the bottom line is defined only up to isomorphism although we can give a very natural explicit model for N, namely it can be represented as the quotient of $B \oplus M$ by the submodule L of elements of the form $(\alpha(a), -f(a))$ for $a \in A$. Then we have $\overline{\beta}(b, m) = \beta(b)$. (Check it is well defined.) It is also useful to have the corresponding formulae for $\overline{\alpha}(m) = (0, m) + L$ and for $\overline{f}(b) = (b, 0) + L$. This gives an extension of modules

$$f^*(\mathcal{E}) : \quad 0 \to M \overset{\overline{\alpha}}{\to} N \overset{\overline{\beta}}{\to} C \to 0.$$

If f extends over B to give g, so $g\alpha = f$, then we have a morphism $g' : N \to M$ given by $g'((m, b) + L) = m + g(b)$. (Check that g' is well defined.)

Lemma 15 *f extends over B if and only if $f^*(\mathcal{E})$ is a split extension.*

Proof: We have done the 'only if'. If $f^*(\mathcal{E})$ is split, there is a projection $g' : N \to M$ such that $g'\overline{\alpha}(m) = m$ for all m. Define $g = g'\overline{f}$ to get the extension. ■

We thus get a map

$$Hom(A, M) \overset{\delta}{\to} Ext^1(C, M)$$

$$\delta(f) = [f^*(\mathcal{E})]$$

which extends the exact sequence one step to the right.

Here it is convenient to define $Ext^1(C, M)$ to be the set (actually Abelian group) of extensions of form

$$0 \to M \to ? \to C \to 0$$

modulo equivalence (isomorphism of middle terms with the ends fixed). The Abelian group structure is given by Baer sum (see entry in Wikipedia, or many standard texts on homological algebra).

Important aside: 'Recall' the '*snake lemma*: given a commutative diagram of modules with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\mu"'] & N \arrow[r] \arrow[d, "\nu"'] & P \arrow[r] \arrow[d, "\psi"'] & 0 \\ 0 \arrow[r] & M' \arrow[r] & N' \arrow[r] & P' \arrow[r] & 0 \end{tikzcd}$$

there is an exact sequence

$$0 \to Ker\,\mu \to Ker\,\nu \to Ker\,\psi \xrightarrow{\delta} Coker\,\mu \to Coker\,\nu \to Coker\,\psi \to 0$$

This has as a corollary that if μ and ψ are isomorphisms then so is ν. (Do check that you can construct δ and prove exactness, i.e. using a simple diagram chase.)

Back to extensions: It is fairly easy to show that $Hom(\mathcal{E}, M)$ extends even further to 6 terms with

$$\ldots \xrightarrow{\beta^*} Ext^1(B, M) \xrightarrow{\alpha^*} Ext^1(A, M)$$

Here is how α^* is constructed. Suppose $\mathcal{E}_1 : 0 \to M \to N \to B \to 0$ gives an element of $Ext^1(B, M)$, then we can form a diagram

$$\begin{tikzcd} \alpha^*(\mathcal{E}_1) : & 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & \alpha^{-1}(N) \arrow[r, "p'"] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\ \mathcal{E}_1 : & 0 \arrow[r] & M \arrow[r] & N \arrow[r, "p"'] & B \arrow[r] & 0 \end{tikzcd}$$

by restricting $\mathcal{E}_1$ along α using a pull back in the right hand square. We can give $\alpha^{-1}(N)$ explicitly in the form that the usual construction of pullbacks in categories of modules gives it to us

$$\alpha^{-1}(N) \cong \{(a, n) \mid \alpha(a) = p(n)\}$$

and p' and α' are projections. The construction of β^* is done similarly using pullback along β. It is then easy to check that the obvious extension to $Hom(\mathcal{E}, M)$, mentioned above, is exact, but that there is again no reason why α^* should be onto. (Of course, knowledge of the purely homological way of getting these exact sequence will suggest that there is an $Ext^2(C, M)$ term to come.)

We examine an obstruction to it being so. Suppose given $\mathcal{E}' : 0 \to M \to N_1 \xrightarrow{p'} A \to 0$, giving us an element of $Ext'(A, M)$. If α^* were onto, we would need a $\mathcal{E}_1 : 0 \to M \to N \to B \to 0$ such that $\alpha^{-1}(N) \cong N_1$ leaving M fixed and relating to α as above by a pullback. We can splice $\mathcal{E}'$ and $\mathcal{E}_1$ together to get a suitable looking diagram

$$\begin{tikzcd} \mathcal{E}' * \mathcal{E}_1 : & 0 \arrow[r] & M \arrow[r] & N' \arrow[rr] \arrow[dr, "p'"'] & & B \arrow[r] & C \arrow[r] & 0 \\ & & & & A \arrow[ur, "\alpha"'] & & & \end{tikzcd}$$

and the row is exact. If we change $\mathcal{E}'$ by an isomorphism than clearly this spliced sequence would react accordingly. If you check up, as suggested, on the Baer sum structure if $Ext^1(A, M)$ and $Ext^2(C, M)$ then you can again check that the above splicing construction yields a homomorphism from the first group to the second. Moreover there is no reason not to extend the splicing construction to a pairing operation on the whole graded family of Ext-groups. This is given in detail in quite a few of the standard books on Homological Algebra, so will not be gone into here.

Two facts we do need to have available are about the structure of $Ext^2(C, M)$. Let $\mathcal{E}xt^2(C, M)$ be the category of 4-term exact sequences

$$0 \to M \to N \to P \to C \to 0$$

and morphisms which are commuting diagrams

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 , \\
0 \arrow[r] & M \arrow[r] & N' \arrow[r] & P' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

then $Ext^2(C, M)$ is the set of connected components of this category. The important thing to note is that the morphisms are not isomorphisms in general, so two 4-term sequences give the same element in $Ext^2(C, M)$ if they are linked by a zig-zag of intermediate terms of this form. The second fact is that the zero for the Baer sum addition is the class of the 4-term extension

$$0 \to M \to M \stackrel{0}{\to} C \to C \to 0$$

with ‘equals’ on the unmarked maps.

Suppose now that the top row in

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N_1 \arrow[r, "\overline{p}"] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r, "p"'] & B \arrow[r] & 0
\end{tikzcd}$$

is obtained by restriction along α from the bottom row. We now form the spliced sequence

$$0 \to M \to N_1 \stackrel{\alpha\overline{p}}{\to} B \to C \to 0.$$

We would hope that this 4-term sequence was trivial, i.e. equivalence to the zero one. We clearly must use the given element in $Ext^1(B, M)$ in a constructive way in the proof that it is trivial, so we form the pushout of $\alpha\overline{p}$ along α' getting us a diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N_1 \arrow[r, "\alpha\overline{p}"] \arrow[d, "\alpha'"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 , \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & B' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

with the middle square a pushout. It is now almost immediate that the morphism from B to B' is split, since we can form a commutative square

$$\begin{tikzcd}
N_1 \arrow[r, "\alpha\overline{p}"] \arrow[d, "\alpha'"'] & B \arrow[d, "="] \\
N \arrow[r, "p"'] & B
\end{tikzcd}$$

giving us the required splitting from B' to B. It is now a simple use of the snake lemma, to show that the complementary summand of B in B' is isomorphic to C. We thus have that the bottom row of the diagram above is of the form

$$0 \to M \to N \to B \oplus C \to C.$$

This looks hopeful but to finish off the argument we just produce the morphism:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & M \arrow[r, "0"] \arrow[d] & C \arrow[r] \arrow[d, "incl_2"] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r, "incl_1 p"'] & B \oplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

and we have a sequence of maps joining our spliced sequence to the trivial one. (A similar argument goes through in higher dimensions.) Now you should try to prove that if a spliced sequence is linked to a trivial one then it does come from an induced one. That is quite tricky, so look it up in a standard text. An alternative approach is to use the homological algebra to get the trivialising element (coboundary or homotopy, depending on your viewpoint) and then to construct the extension from that. Another thing to do is to consider how the Ext-groups, $Ext^k(A, M)$, vary in M rather than with A. This will be left to you.

3.8.3 From Ext to group cohomology

If we look briefly at the classical homological algebraic method of defining $Ext^K(A, M)$, we would take a projective resolution $\mathsf{P}.$ of A, apply the functor $Hom(-, M)$, to get a cochain complex $Hom(\mathsf{P}., M)$, then take its (co)homology, with $H^n(Hom(\mathsf{P}., M))$ being isomorphic to $Ext^n(A, M)$, or, if you prefer, $Ext^n(A, M)$ being defined to be $H^n(Hom(\mathsf{P}., M))$. This method can be studied in most books on homological algebra (we cite for instance, MacLane, [92], Hilton and Stammbach, [72] and Weibel, [135]), so is easily accessible to the reader - and we will not devote much space to it here as a result. We will however summarise some points, notation, definitions of terms etc., some of which you probably know.

First the notion of projective module:

Definition: A module P is *projective* if, given any epimorphism, $f : B \to C$, the induced map $Hom(P, f) : Hom(P, B) \to Hom(P, C)$ is onto. In other words any map from P to C can be lifted to one from P to B.

Any free module is projective.

Of the properties of projectives that we will use, we will note that $Ext^n(P, M) = 0$ for P projective and for any M. To see this recall that any n-fold extension of P by M will end with an epimorphism to P, but such things split as their codomain is projective. It is now relatively easy to use this splitting to show the extension is equivalent to the trivial one.

A resolution of a module A is an augmented chain complex

$$\mathsf{P}. : \ldots \to P_1 \to P_0 \to M$$

which is exact, i.e. it has zero homology in all dimensions. This means that the augmentation induces an isomorphism between $P_0/\partial P_1$ and M. The resolution is projective if each P_n is a projective module.

If $\mathsf{P}.$ and $\mathsf{Q}.$ are both projective resolutions of A, then the cochain complexes $Hom(\mathsf{P}., M)$ and $Hom(\mathsf{Q}., M)$ always have the same homology. (Once again this is standard material from homological algebra so is left to the reader to find in the usual sources.)

An example of a projective resolution is given by the bar resolution, $\mathsf{B}G.$, and the construction $C^n(G, M)$ in the first chaper is exactly $Hom(\mathsf{B}G., M)$. This reolution ends with $BG_0 = \mathbb{Z}[G]$ and the resolution resolves the Abelian group $\mathbb{Z}$ with trivial G-module structure. (This can be seen from our discussion of homological syzygies where we had

$$\mathbb{Z}[G]^{(R)} \to \mathbb{Z}[G]^{(X)} \to \mathbb{Z}[G] \to \mathbb{Z}.$$

In fact we have

$$H^n(G, M) \cong Ext^n(\mathbb{Z}, M)$$

by the fact that $\mathsf{B}G.$ is a projective resolution of $\mathbb{Z}$ and then we can get more information using the short exact sequence

$$0 \to I(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

As $\mathbb{Z}[G]$ is a free G-module, it is projective and the long exact sequence for $Ext(-, M)$ thus has every third term trivial (at least for $n > 0$), so

$$Ext^n(\mathbb{Z}, M) \cong Ext^{n-1}(I(G), M)$$

giving another useful interpretation of $H^n(G, M)$.

3.8.4 Exact sequences in cohomology

Of course, the identification of $H^n(G, M)$ as $Ext^n(\mathbb{Z}, M)$ means that, if

$$0 \to L \to M \to N \to 0$$

is an exact sequence of G-modules, we will get a long exact sequence in $H^n(G, -)$, just by looking at the long exact sequence for $Ext^n(\mathbb{Z}, -)$.

What is more interesting - but much more difficult - is to study the way that $H^n(G, M)$ varies as G changes. For a start it is not completely clear what this means! If we change the group in a short exact sequence,t

$$1 \to G \to H \to K \to 1$$

say, then what type of modules should be used fro the 'coefficients', that is to say a G-modules or one over H or K. This problem is, of course, related to the change of groups along an arbitrary homomorphism, so we will look at an group homomorphism $\varphi : G \to H$, with no assumptions as to monomorphism, or normal inclusion, at least to start with.

Suppose given such a φ, then the 'restriction functor' is

$$\varphi^* : H{-}Mod \to G{-}Mod,$$

where, if N is in $H{-}Mod$, $\varphi^*(N)$ has the same underlying Abelian group structure as N, but is a G-module via the action, $g.n := \varphi(g).n$. We have already used that φ^* has a left adjoint φ_* given by $\varphi_*(M) = \mathbb{Z}H \otimes_{\mathbb{Z}G} M$. Now we also need a right adjoint for φ^*.

To construct such an adjoint, we use the old device of assuming that it exists, studying it and then extracting a construction from that study. We have M in $G{-}Mod$ and N in $H{-}Mod$, and we assume a natural isomorphism

$$G{-}Mod(\varphi^*(N), M) \cong H{-}Mod(N, \varphi_\sharp(M)).$$

If we take $N = \mathbb{Z}H$, then, as $H{-}Mod(\mathbb{Z}H, \varphi_\sharp(M)) \cong \varphi_\sharp(M)$, we have a construction of $\varphi_\sharp(M)$, at least as an Abelian group. In fact this gives

$$\varphi_\sharp(M) \cong G{-}Mod(\varphi^*(\mathbb{Z}H), M)$$

and as $\mathbb{Z}H$ is also a right G-module, via $h.g := h.\varphi(g)$, we have a left G-module structure of $\varphi_\sharp(M)$ as expected. In fact, this is immediate from the naturality of the adjunction isomorphism using the left hand position of $G{-}Mod(\varphi^*(\mathbb{Z}H), M)$, as for fixed M, the functor converts the right G-action of $\mathbb{Z}$ to a left one on $\varphi_\sharp(M)$. This allows us to get an explicit elementwise formula for this action as follows: let $m^* : \mathbb{Z}H \to M$ be a left G-module mrphsim This can be specified by what it does to the natural basis of $\mathbb{Z}H$ (as Abelian group), and so is often written $m^* : H \to M$, where the function m^* must satisfy a G-equivariance property: $m^*(\varphi(g).h) = g.m^*(h)$. Any such function can, of course, be extended linearly to a G-module morphism of the earlier form. If $g \in G$, we get a morphism

$$-.\varphi(g) : \varphi^*(\mathbb{Z}H) \to \varphi^*(\mathbb{Z}H)$$

given by 'h goes to $h\varphi(g)$'. This *is* a G-module morphism as the G-module structure is by *left* multiplication, which is independent of this right multiplication. Applying $G{-}Mod(-, M)$, we get $g.m^*$ is given by

$$g.m^*(h) - m^*(h.\varphi(g).$$

This is a *left* G-module structure, although at first that may seem strange. That it is linear is easy to check. What take a little bit of work is to check $(g_1g_2).m^* = g_1(g_2.m^*)$: applying both sides to an element $h \in H$ gives

$$(g_1g_2).m^*(h) = m^*(h\varphi(g_1)\varphi(g_2)),$$

whilst

$$g_1(g_2.m^*)(h) = (g_2.m^*)(h.\varphi(g_1)) = m^*(h\varphi(g_1)\varphi(g_2)).$$

(The checking that $g_1.m^*$ does satisfy the G-equivariance property is left to the reader.)

Remark: There are great similarities between the above calculations and those needed later when examining bitorsors. This is certainly not coincidental!

We built $\varphi_\sharp(M)$ in such a way that it is obviously functorial in M and gives a right adjoint to φ^*. This implies that there is a natural morphism

$$i : N \to \varphi_\sharp\varphi^*(N).$$

We denote this second module by N^*, when the context removes any ambiguity, and especially when φ is the inclusion of a subgroup. The morphism sends n to $n^* : H \to N$, where $n^*(h) = h.n$. (Check that $n^*(\varphi(g).h) = g.n^*(h)$. This reminds us that the codomain of n^* is infact just the *set* N underlying both the H-module N and the G-module $\varphi^*(N)$.)

We examine the cohomology groups $H^n(H, N^*)$. These are the (co)homology groups of the cochain complex $Hom(\mathsf{P}., N^*)$, where $\mathsf{P}.$ is a projective H-module resolution of $\mathbb{Z}$. The adjunction shows that this is isomorphic to $Hom(\varphi^*(\mathsf{P}.), \varphi^*(N))$. If $\varphi^*(\mathsf{P}.)$ is a projective G-module resolution of the trivial G-module $\mathbb{Z}$ then the cohomology of this complex will be $H^n(G, N)$, where N has the structure $\varphi^*(N)$.

The condition that free or projective H modules restrict to free or projective G-modules is satisfied in one important case, namely when G is a subgroup of H, since $\mathbb{Z}H$ is a free Abelian group on the *set* H and H is a disjoint union of right G-cosets, so $\mathbb{Z}H$ splits as a G-module into a direct sum of copies of $\mathbb{Z}G$. This provides part of the proof of Shapiro's lemma

Proposition 19 *If $\varphi : G \to H$ is an inclusion, then for a H-module N, there is a natural isomorphism*

$$H^n(H, N^*) \cong H^n(G, N).$$

■

Corollary 4 *The morphism $i : N \to N^*$ and the above isomorphism yield the restriction morphism*

$$H^n(H, N) \to H^n(G, N).$$

■

This suggest other results. Suppose we have an extension

$$1 \to N \to G \to Q \to 1$$

(so here we replace H by G with N in the old role of G, but in addition, being normal in G).

If we look at $\mathsf{B}N$ and $\mathsf{B}G$ in dimension n, these are free modules over the sets N^n and G^n respectively, with the inclusion between them; G is a disjoint union of N-cosets, indexed by elements of Q, so can we use this to derive properties of the cokernel of $\mathbb{Z}G \otimes_{\mathbb{Z}N} \mathsf{B}N \to \mathsf{B}G$, and to tie them into some resolution of Q, or perhaps, of $\mathbb{Z}$ as a trivial Q-module. The answer must clearly be positive, perhaps with some restrictions such as finiteness, but there are several possible ways of getting to an answer having slightly different results. (You have in the (φ_*, φ^*) and $(\varphi^*, \varphi_\sharp)$ adjunctions, enough of the tools needed to read detailed accounts in the literature, so we will not give them here.)

This also leads to relative cohomology groups and their relationship with the cohomology of the quotient Q. We can also consider the crossed resolutions of the various groups in the extension and work, say, with the induced maps

$$\mathsf{C}(N) \to \mathsf{C}(C)$$

looking at its cokernel or better what should be called its homotopy cokernel.

Another possibility is to examine $\mathsf{C}(N)$ and $\mathsf{C}(Q)$ and the cocycle information needed to specify the extension, and to use all this to try to construct a crossed resolution of G. (We will see something related to this in our examination of non-Abelian cohomology a little later.) A simple case of this is when the extension is split, $G \cong N \rtimes Q$ and using a twisted tensor product for crossed complexes, one can produce a suitable $\mathsf{C}(N) \otimes_\tau \mathsf{C}(Q)$ resolving G, (see Tonks, [127]).

Chapter 4

Syzygies, and higher generation by subgroups

Syzygies are one of the routes to working with resolutions. They often provide insight as to how a presentation relates to geometric aspects of a group, for instance giving structured spaces such as simplicial complexes, or, better, polytopes, on which the group acts. Syzygies extend the role of 'relations' in group presentations to higher dimensions and hence are 'relations between relations ... between relations'. They thus form a very well structured (and thus simpler) case of higher dimensional rewriting. Later we will see relations between this and several important aspects of cohomology. We will also explore some links with ideas from rewriting theory.

4.1 Back to syzygies

There are both homotopical and homological syzygies. We have met homological syzygies earlier and also have:

4.1.1 Homotopical syzygies

We have built a complex, $K(\mathcal{P})$, from a presentation, $\mathcal{P}$, of a group, G. Any element in $\pi_2(K(\mathcal{P}))$ can, of course, be represented by a map from S^2 to $K(\mathcal{P})$ and, by cellular approximation, can be replaced, up to homotopy, by a cellular decomposition of S^2 and a cellular map $\phi : S^2 \to K(\mathcal{P})$. We will adopt the terminology of Kapranov and Saito, [84], and Loday, [90], and say

Definition: A *homotopical 2-syzygy* consists of a cellular subdivision of S^2 together with a map, $\phi : S^2 \to K(\mathcal{P})$, cellular for that decomposition..

Of course, such an object corresponds to an identity among the relations of $\mathcal{P}$, but is a *specific representative* of such an identity. The specification of the cellular decomposition provides valuable combinatorial and geometric information on the presentation.

Definition: A family, $\{\phi_\lambda\}_{\lambda\in\Lambda}$, of such homotopical 2-syzygies is then called *complete* when the homotopy classes $\{[\phi_\lambda]\}_{\lambda\in\Lambda}$ generate $\pi_2(K(\mathcal{P}))$.

In this case, we can use the ϕ_λ to form the next stage of the construction of an Eilenberg-Mac Lane space, $K(G,1)$, by killing this π_2. More exactly, rename $K(\mathcal{P})$ as $X(2)$ and form

$$X(3) := X(2) \cup \bigcup_{\lambda\in\Lambda} e^3_\lambda,$$

by, for each $\lambda \in \Lambda$, attaching a 3-cell, e^3_λ, to $X(2)$ using ϕ_λ. Of course, we then have

$$\pi_1(X(3)) \cong G, \quad \pi_2(X(3)) = 0.$$

Again $\pi_3(X(3))$ may be non-trivial, so we consider *homotopical 3-syzygies.* Such a thing, s, will consist of an oriented polytope decomposition of S^3 together with a continuous map, f_s from S^3 to $X(3)$, which sends the i-skeleton of that decomposition to $X(i)$, $i = 0, 1, 2$.

At this stage we have $X(0) = K(\mathcal{P})_0$, a point, $X(1) = K(\mathcal{P})_1$, and $X(2) = K(\mathcal{P})_2$. One wants enough such 3-syzygies, s, identified algebraically and combinatorially, so that the corresponding homotopy classes, $\{[f_s]\}$ generate $\pi_3(X(3))$.

It is clear, by induction, we get a notion of *homotopical n-syzygy.* We assume $X(n)$ has been built inductively by attaching cells of dimension $\leq n$ along homotopical k-syzygies for $k < n$, so that

$$\pi_1(X(n)) \cong G, \quad \pi_k(X(n)) = 0, \quad k = 2, \ldots, n-1,$$

then a *homotopical n-syzygy*, s, is an oriented polytope decomposition of S^n and a continuous cellular map $f_s : S^n \to X(n)$.

After a choice of a set, $\mathcal{R}_n$, of n-syzygies, so that $\{[s_s] \mid s \in \mathcal{R}_n\}$ generates $\pi_n(X(n))$ as a G-module, we can form $X(n+1)$ by attaching $n+1$-dimensional cells e^{n+1}_s along these f_s for $s \in \mathcal{R}_n$.

If we can do this in a sensible way, for all n, we say the resulting system of syzygies is *complete* and the limit space $X(\infty) = \bigcup X(n)$ is then a cellular model for BG, the classifying space of the group G. We will look at classifying spaces again later.

This construction is, of course, just a homotopical version of the construction of a free resolution of the trivial G-module, $\mathbb{Z}$.

Remark: Some additional aspects of this can be found in Loday's paper [90], in particular the link with the 'pictures' of Igusa, [78, 79].

Example and construction: Given any group, G, we can find a presentation with $\{\langle g\rangle \mid g \neq 1, g \in G\}$ as set of generators and a relation, $r_{g,g'} := \langle g\rangle\langle g'\rangle\langle g'g\rangle^{-1}$, for each pair (g, g') of elements of G. (We write $\langle 1\rangle = 1$ for convenience.) We will have earlier call this the *standard presentation* of the group, G. It is closely related to the nerve of $G[1]$, and also to the various bar resolutions. (There may be a need later to consider a variant in which the identity element of G is not excluded as a generator, however that will still be loosely called the standard presentation. Note that since then $\langle 1\rangle.\langle g\rangle = \langle 1.g\rangle = \langle g\rangle$, the identification $\langle 1\rangle = 1$ is automatic.)

The relation $r_{g,g'}$ gives a triangle

$$\begin{tikzcd}
& \cdot \arrow[dr, "g'"] & \\
\cdot \arrow[ur, "g"] \arrow[rr, "g'g"'] & & \cdot
\end{tikzcd}$$

and, for each triple (g, g', g''), we get a homotopical 2-syzygy in the form of a tetrahedron.

Higher homotopical syzygies occur for any tuple, $(g_1, \ldots, g_n)$, of non-identity elements of G, by labelling a n-simplex. The limiting cellular space, $X(\infty)$, constructed from this context is just the usual model of the classifying space, BG, as geometric realisation of the *nerve* of G, or if you prefer, of the groupoid $G[1]$ with one object. The corresponding free resolution, $(C_*(G), d)$, is the classical *normalised bar resolution*. Using the bar resolution above dimension 2 together with the crossed module of the presentation at the base, one gets the standard free crossed resolution of the group, G, as we saw in section 3.1.2.

4.1.2 Syzygies for the Steinberg group

(This is adapted from Kapranov and Saito, [84].)

Let R be an associative ring with 1. Recall that the (n^{th} unstable) *Steinberg group*, $St_n(R)$, has generators, $x_{ij}(a)$, labelling the elementary matrices $\varepsilon_{ij}(a)$, having

$$\varepsilon_{ij}(a)_{k,l} = \begin{cases} 1 & \text{if } k = l \\ a & \text{if } (k,l) = (i,j), a \in R \\ 0 & \text{otherwise,} \end{cases}$$

and relations

St1 $\quad x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a+b);$

St2 $\quad [x_{i,j}(a), x_{k,\ell}(b)] = \begin{cases} 1 & \text{if } i \neq \ell, j \neq k, \\ x_{i,\ell}(ab) & i \neq \ell, j = k \end{cases}$ and in which all indices are positive integers less than or equal to n.

The terminology 'n^{th} unstable' is to make the contrast with the group $St(R)$, the stable version. The unstable version, $St_n(R)$, models 'universal' relations satisfied by the $n \times n$ elementary matrices, whilst, in $St(R)$, the indices, i, j, k etc. are not constrained to be less than or equal to n. We will look at the stable version later.

The identities / homotopical 2-syzygies are built from three types of polygon:

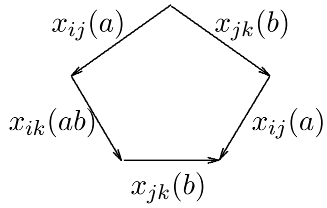
a) a triangle, $T_{ij}(a,b)$ for each i, j, $i \neq j$, coming from St1;

b) a square,

$$\begin{tikzcd} \cdot \arrow[r, "x_{ij}(a)"] \arrow[d, "x_{kl}(b)"'] & \cdot \arrow[d, "x_{kl}(b)"] \\ \cdot \arrow[r, "x_{ij}(a)"'] & \cdot \end{tikzcd}$$

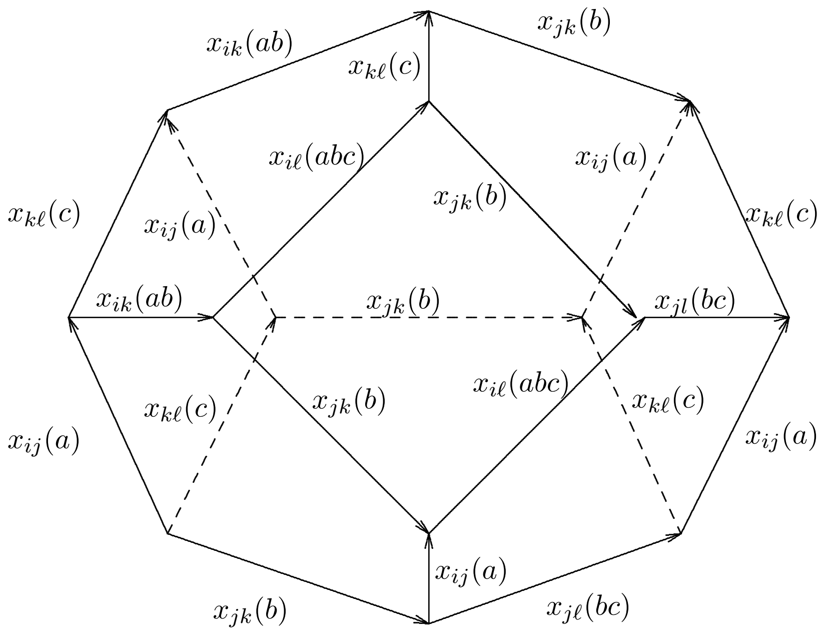
corresponding to the first case of St2 and

c) a pentagon, for the second:



Then for any pairs (i, j), (k, l), (m, p) with $x_{ij}(a)$, $x_{kl}(b)$, $x_{mp}(c)$, commuting by virtue of St2's first clause, we will have a homotopical syzygy in the form of a labelled cube.

There is also a homotopy 2-syzygy given by the associahedron labelled by generators as shown:



Remark: Kapranov and Saito, [84], have conjectured that the space $X(\infty)$ obtained by gluing labelled higher Stasheff polytopes together, is homotopically equivalent to the homotopy fibre of

$$f : BSt(R) \to BSt(A)^+,$$

where $(-)^+$ denotes Quillen's plus construction. The associahedron is a Stasheff polytope and, by encoding the data that goes to build the identities / syzygies schematically in a 'hieroglyph', Kapranov and Saito make a link between such hieroglyphs and polytopes.

4.2 A brief sideways glance: simple homotopy and algebraic K-theory

The study of the Steinberg group is closely bound up with the development of algebraic K-theory. That subject grew out of two apparently unrelated areas of algebraic geometry and algebraic topology. The second of these, historically, was the development by Grothendieck of (geometric and topological) K-theory based on projective modules over a ring, or finite dimensional vector bundles on a space. (The connection between these is that the space of global sections of a finite dimensional vector bundle on a nice enough space, X, is a finitely generated projective module over the ring of continuous real or complex functions on X. We will look at vector bundles and this link with K-theory a bit more in detail later on; see section **??**. We will be discussing other forms of K-theory in that section as well, so will not give more detail on that more purely topological side of the subject here.)

Algebraic K-theory was initially a body of theory that attempted to generalise parts of linear algebra, notably the theory of dimension of vector spaces, and determinants to modules over arbitrary rings. It has grown into a well developed tool for studying a wide range of algebraic, geometric and even analytic situations from a variety of points of view.

For the purposes here we will give a short description of the low dimensional K-groups of a ring, R, with for initial aim to provide examples for use with the further discussion of rewriting, group presentations, syzygies, and homotopy. The discussion will, however, also look a bit more deeply at various other aspects when they seem to fit well into the overall structure of the notes.

4.2.1 Grothendieck's $K_0(R)$

For our discussion here, it will suffice to say that, given an associative ring, R, we can form the set, $[Proj_{fg}(R)]$ of isomorphism classes of finitely generated projective modules over R. Direct sum gives this a monoid structure. This is then 'completed' to get an Abelian group. We will give a more detailed discussion of this later in Proposition **??**, but here we will just give the formula:

$$K_0(R) := F([Proj_{fg}(R)])/\langle [P] + [Q] - [P \oplus Q] \rangle$$

in which P and Q are finitely generated projective modules, F is the free Abelian group functor and $[P]$ indicates the isomorphism class of P. The relations force the abstract addition in the free Abelian group to mirror the direct sum induced addition on the generators.

4.2.2 Simple homotopy theory

The other area that led to algebraic K-theory was that of simple homotopy theory. J. H. C. Whitehead, following on from earlier ideas of Reidemeister, looked at possible extensions of combinatorial group theory, with its study of presentations of groups, to give a combinatorial homotopy theory; see [139]. This would take the form of an 'algebraic homotopy theory' giving good algebraic models for homotopy types, and would hopefully ease the determination of homotopy equivalences for instance of polyhedra. The 'combinatorial' part was exemplified by his two papers on 'Combinatorial Homotopy Theory' [137, 138], but raised an interesting question. In combinatorial group theory, a major role is played by Tietze's theorem:

Theorem 6 *(Tietze's theorem, 1908, [123]) Given two finite presentations of the same group, one can be obtained from the other by a finite sequence of Tietze transformations.* ■

Proofs of this are easy to find in the literature. For instance, one based on a series of exercises is given in Gilbert and Porter, [65], p.135.

We clearly need to make precise what are the Tietze transformations.

Let $\mathcal{P} = (X : R)$ be a group presentation of a group, G and set $F(X)$ to be the free group on the set X. We consider the following transformations:

T1: Adding a superfluous relation: $(X : R)$ becomes $(X : R')$, where $R' = R \cup \{r\}$ and $r \in N(R)$, the normal closure of the relations in the free group on X, i.e., r is a consequence of R;

T2: Removing a superfluous relation: $(X : R)$ becomes $(X : R')$ where $R' = R - \{r\}$, and r is a consequence of R';

T3: Adding a superfluous generator: $(X : R)$ becomes $(X' : R')$, where $X' = X \cup \{g\}$, g being a new symbol not in X, and $R' = R \cup \{wg^{-1}\}$, where w is a word in the other generators, that is w is in the image of the inclusion of $F(X)$ into $F(X')$;

T4: Removing a superfluous generator: $(X : R)$ becomes $(X' : R')$, where $X' = X - \{g\}$, and $R' = R - \{wg^{-1}\}$ with $w \in F(X')$ and $wg^{-1} \in R$ and no other members of R' involve g.

Definition: These transformations are called *Tietze transformations.*

The question was to ask if there was a higher dimensional version of the Tietze transformations that would somehow generate all *homotopy equivalences.*

Let us imagine the transformation of the complex, $K(\mathcal{P})$, of $\mathcal{P}$ under these moves. The complex is, of course, a simple form of CW-complex, built by attaching cells in dimensions 1 and then 2. If we add a superfluous generator to $\mathcal{P}$ as above (T3), then effectively we add a 2-cell labelled by wg^{-1} and it will be glued on by an attaching map that is defined on a semi-circle in its boundary and on which the path represents the word, w. The other semi-circle yields the loop representing the new generator. This process therefore does not change the homotopy type of $K(\mathcal{P})$. On the other hand, adding a superfluous relation will change the homotopy type of the complex. The new relation corresponds to a 2-cell glued on to $K(\mathcal{P})$, but the attaching map is already null-homotopic in $K(\mathcal{P})$ as it represents a consequence of the relations. The effect is that $K(\mathcal{P}')$ has the homotopy type of $K(\mathcal{P}) \vee S^2$, and the module of identities has an extra free summand.

These thus show both types of behaviour when attaching a cell to a pre-existing complex. In the first, the relation 2-cell is attached by part of its boundary. In the second the new cell is attached by gluing along *all* of its boundary, so will change the homotopy type of $K(\mathcal{P})$. It will not change its fundamental group, just its higher homotopy groups. This raises and interesting question, and that is to mirror these Tietze transformations by higher order ones which do not change the n-type, for some n, but may change the whole homotopy type, but we need to get back towards simple homotopy theory.

Tietze transformations had given a way of manipulating presentations and thus suggested a way of manipulating complexes. The thought behind simple homotopy theory was to produce a way of constructing homotopy equivalences between complexes. This, if it worked, might simplify the task of determining whether two spaces (defined, say, as simplicial complexes) were of the same homotopy type, and if so was it possible to build up the homotopy equivalences between them in some simple way.

The resulting theory was developed initially by Reidemeister and then by Whitehead, culminating in his 1950 paper, [140]. The theory received a further important stimulus with Milnor's classic paper, [98], in which the emphasis was put on elementary expansions.

(A good source for the theory of simple homotopy is Cohen's book, [40].)

We will work here with finite CW-complexes. These are built up by induction by gluing on n-cells, that is copies of $D^n = \{x \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}$, at each stage. Each D^n has a boundary an $(n-1)$-sphere, $S^{n-1} = \{x \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$. The construction of objects in the category of finite CW-complexes is by attaching cells by means of maps defined on part of all of the boundary of a cell. This will usually change the homotopy type of the space, creating or filling in a 'hole'. The homotopy type will not be changed if the attaching map has domain a hemisphere. We write $S^{n-1} = D_-^{n-1} \cup D_+^{n-1}$, with each hemisphere homeomorphic to a $(n-1)$-cell, and their intersection being the equatorial $(n-2)$-sphere, S^{n-2}, of S^{n-1}.

Given, now, a finite CW-complex, X, we can build a new complex Y, consisting of X and two new cells, e^n and e^{n-1} together with a continuous map, $\varphi : D^n \to Y$ satisfying
(i) $\varphi(D_+^{n-1}) \subseteq X_{n-1}$;
(ii) $\varphi(S^{n-2}) \subseteq X_{n-2}$;
(iii) the restriction of φ to the interior of D^n is a homeomorphism onto e^n;
and
(iv) the restriction of φ to the interior of D_-^{n-1} is a homeomorphism onto e^{n-1}.

There is an obvious inclusion map, $i : X \to Y$, which is called an *elementary expansion*. There is also a retraction map $r : Y \to X$, homotopy inverse to i, and which is called an *elementary contraction*. Both are homotopy equivalences. Can all homotopy equivalences between finite CW-complexes be built by composing such elementary ones? More precisely if we have a homotopy equivalence $f : X \to X'$, is f homotopic to a composite of a finite sequence of elementary expansions and contractions? Such a homotopy equivalence would be called *simple*. Whitehead showed that not all homotopy equivalences are simple and constructed a group of obstructions for the problem with given space X, each non-identity element of the group corresponding to a distinct homotopy class of non-simple homotopy equivalences.

4.2.3 The Whitehead group and $K_1(R)$

We will very briefly sketch how the investigation goes, skimming over the details; for them, see Milnor, [98], or Cohen's book, [40].

Starting with a homotopy equivalence, $f : X \to Y$, we can convert it to a deformation retraction using the mapping cylinder construction. (We will see this in more detail later, but do not need that detail here). This means that we have a CW-pair, (Y, X), with a deformation retraction from Y to X. Classifying the simple homotopy types of X is then transformed into a problem of classifying these. Passing first to their universal covering spaces, $\tilde{Y}$ and $\tilde{X}$, and then to the cellular chain complexes associated to both these, the problem is reduced to examining the *relative* cellular chain complex, $C(\tilde{Y}, \tilde{X})$, obtained from the exact sequence

$$0 \to C(\tilde{X}) \to C(\tilde{Y}) \to C(\tilde{Y}, \tilde{X}) \to 0$$

All of these can be considered as chain complexes of modules over the group ring of $\pi_1 X$. As there are only finitely many cells in X and Y, this chain complex has only finitely many non-zero levels in it. It is also acyclic, i.e., has zero homology because the inclusion of $C(\tilde{X})$ into $C(\tilde{Y})$ induces isomorphism on homology. The cells in $Y - X$ give a preferred basis to the modules concerned.

One further reduction takes the direct sum of the even dimensional $C(\tilde{Y}, \tilde{X})_n$, and similarly that of the odd ones, and the induced boundary from the odds to the evens. (At each stage the reduction is checked to preserve what one want, namely whether or not the inclusion of X into Y is given by some combinations of elementary expansions and contractions. (The last part of this can be examined intuitively by thinking about what happens if you add in an n-cell by a $n-1$-cell in its boundary.)

This reduces the task to one of examining an isomorphism between two based free modules over $\mathbb{Z}\pi_1 X$, and that brings us, finally, to the main point of this section namely the definition of the group $K_1(R)$. (For this original application to simple homotopy theory, one takes $R = \mathbb{Z}\pi_1 X$.)

We will not take a historical order, concentrating on K_1, which was extracted from Whitehead's work, and studied for its own sake by Bass, [12]. Other aspects relating to simple homotopy theory may be looked at later on when we have more tools available.

Let R be an associative ring with 1. As usual $G\ell_n(R)$ will denote the *general linear group* of $n \times n$ non-singular matrices over R. There is an embedding of $G\ell_n(R)$ into $G\ell_{n+1}(R)$ sending a matrix $M = (m_{i,j})$ to the matrix M' obtained from M by adding an extra row and columnof zeros except that $m'_{n+1,n+1} = 1$. This gives a nested sequence of groups

$$G\ell_1(R) \subset G\ell_2(R) \subset \ldots \subset G\ell_n(R) \subset G\ell_{n+1}(R) \subset \ldots$$

and we write $G\ell(R)$ for the colimit (union) of these. It will be called the *stable general linear group over R*

Definition: The group, $K_1(R)$, is $G\ell(R)^{Ab} = G\ell(R)/[G\ell(R), G\ell(R)]$.

This is functorial in R, so that a ring homomorphism, $\varphi : R \to S$ induces $K_1(\varphi) : K_1(R) \to K_1(S)$.

The main initial problem with the above definition of $K_1(R)$ is that of controlling the commutator subgroup of $G\ell(R)$. The key is the *stable elementary linear group*, $E(R)$.

We extend the earlier definition of elementary matrices (on page 115 from the finite dimensional case, i.e., within $G\ell_n(R)$, to being within $G\ell(R)$. Here an *elementary matrix* is of the form $e_{ij}(a) \in G\ell(R)$, for some pair (i, j) of distinct positive integers and which, thus, has an a in the (i, j) position, 1s in every diagonal position and 0 elsewhere. Although there is a small risk of confusion from notational reuse, we will, none-the-less, follow the standard notational convention and write $E_n(R)$ for the subgroup generated by the elementary matrices in $G\ell_n(R)$ and $E(R)$ for the corresponding union of the $E_n(R)$ within $G\ell(R)$. We will call $E_n(R)$ the *elementary subgroup of $G\ell_n(R)$*,

Lemma 16 *If i, j, k are distinct positive integers, then*

$$e_{ij}(a) = [e_{ik}(a), e_{kj}(1)].$$

■

This was already commented on when looking at the Steinberg group, $St_n(R)$, which abstracts the 'generic' properties of the elementary matrices. The following is now obvious.

Proposition 20 *For $n \geq 3$, $E_n(R)$ is a perfect group, i.e.,*

$$[E_n(R), E_n(R)] = E_n(R).$$

■

Now let $M = (m_{ij})$ be any $n \times n$ matrix over R. (It is not assumed to be invertible.)

We note that in $G\ell_{2n}(R)$,

$$\begin{pmatrix} I_n & M \\ 0 & I_n \end{pmatrix} = \prod_{i=1}^{n} \prod_{j=1}^{n} e_{i,j+n}(m_{ij}),$$

so this is in $E_{2n}(R)$. Similarly $\begin{pmatrix} I_n & 0 \\ M & I_n \end{pmatrix} \in E_{2n}(R)$.

Next, let $M \in G\ell_n(R)$ and note

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ M^{-1} - I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ M - I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & -M^{-1} \\ 0 & I_n \end{pmatrix}$$

(as is easily verified). We thus have

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \in E_{2n}(R),$$

hence it is a product of commutators.

Lemma 17 *If $M, N \in G\ell_n(R)$, then*

$$\begin{pmatrix} [M,N] & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & N^{-1} \end{pmatrix} \begin{pmatrix} (NM)^{-1} & 0 \\ 0 & NM \end{pmatrix},$$

so is in $E_{2n}(R)$.

Proof: Just calculation. ■

Passing to the stable groups, we get the famous *Whitehead lemma*:

Proposition 21

$$[G\ell(R), G\ell(R)] = E(R).$$

■

This was, thus, very easy to prove, but it is crucial for the development of algebraic K-theory. It should be noted that it did depend on having 'enough dimensions', so $[G\ell_n(R), G\ell_n(R)] \subseteq E_{2n}(R)$. For our purposes here, we do not need to question whether 'unstable' versions of this hold, however we will mention that, if $n \geq 3$ and R is a commutative ring, then $[G\ell_n(R), G\ell_n(R)] = E_n(R)$. The proof is given in many texts on algebraic K-theory.

4.2.4 Milnor's K_2

We have already met the definition of $K_2(R)$ (page 37). The stable elementary linear group, $E(R)$, is a quotient of the stable Steinberg group, $St(R)$. (It will help to glance back at the presentation given on page 99 and to check that these are 'generic' relationships between elementary matrices.) This stable Steinberg group is obtained from the various $St_n(R)$ together with the inclusions $St_n(R) \to ST_{n+1}(R)$ obtained by including the generators of the first into the generating set of the second in the obvious way. the colimit of these 'unstable' groups yields the *stable Steinberg group*

As we mentioned early and will prove shortly, there is a central extension:

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

and thus $\varphi : St(R) \to E(R)$, a crossed module. The group, $G\ell(R)/Im(b)$, is $K_1(R)$, the first algebraic K-group of the ring.

In fact, this is a universal central extension and certain observations about such objects will help interpret what information is contained in $K_2(R)$. We will 'backtrack' a bit so as to keep things relatively self-contained.

Let, as usual, $Z(G)$ denote the centre of a group G.

Lemma 18 *(i)* $Z(E(R)) = 1$;
(ii) $Z(St(R)) = K_2(R)$.

Proof: This is elementary, but fun!

Suppose that $N \in Z(E(R))$, then $N \in E_n(R)$ for some n. Within $E_{2n}(R)$,

$$\begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}\begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix},$$

since N is central in $E(R)$. This works out as

$$\begin{pmatrix} N & N \\ 0 & I \end{pmatrix} = \begin{pmatrix} N & I \\ 0 & I \end{pmatrix},$$

i.e., $N = I$.

Next suppose that $M \in Z(St(R))$, then, as φ is surjective, $\varphi(M) \in Z(E(R))$, so must be trivial, as required. ■

Proposition 22

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

is a central extension. ■

We next need to examine *universal* central extensions.

Definitions: (i) A central extension

$$1 \to K \xrightarrow{k} H \xrightarrow{\sigma} G \to 1$$

is said to be *weakly universal* if, given any other central extension of G,

$$1 \to L \xrightarrow{k'} E \xrightarrow{\sigma'} G \to 1,$$

there is a homomorphism $\psi : H \to E$ making the diagram

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "k"] \arrow[d, "\psi|_K"'] & H \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & L \arrow[r, "k'"'] & E \arrow[r, "\sigma'"'] & G \arrow[r] & 1
\end{tikzcd}$$

commutes.

(ii) The central extension, as above, of G is *universal* if it is weakly universal and, in the previous definition, the morphism ψ is unique with that property.

Proposition 23 *Every group has a weakly universal central extension.*

Proof: Suppose that we have a presentation $(X : R)$ of G, or more usefully for us, a presentation sequence:

$$1 \to K \overset{k}{\to} F \overset{p}{\to} G \to 1,$$

(so $F = F(X)$, the free group on X, and $K = N(R)$ is the kernel of p). The subgroup, $[K, F]$. of F generated by the commutators, $[k(x), y]$, with $x \in K$, and $y \in F$, is normal , as is easily checked and is in K, so we can form an extension

$$1 \to \frac{K}{[K,F]} \to \frac{F}{[K,F]} \to G \to 1.$$

(Note that 'dividing out by this subgroup identifies all $k(x)y$ and $yk(x)$, so should make a central extension. It 'kills' the conjugation action of F on K.)

We will write $H = F/[K, F]$ with $\sigma : H \to G$ for the induced epimorphism, so we now have

$$\mathbb{E} : 1 \to Ker\,\sigma \to H \overset{\sigma}{\to} G \to 1.$$

This is a central extension, as is easily checked (**left to you**).

Now suppose

$$\mathbb{E}' : 1 \to L \overset{k}{\to} E \overset{\sigma'}{\to} G \to 1$$

is another central extension. We have to construct a morphism, $\psi : \mathbb{E} \to \mathbb{E}'$, i.e., $\varphi : H \to E$, compatibly with the projections to G, (and their kernels). As F is free and σ' is an epimorphism, we can find $\tau : F \to E$ such that $\sigma\tau = p$. Now $\sigma'\tau k = 1$, so $\tau k = k'\psi|_K : K \to L$. We examine a commutator $[k(x), y]$ with $x \in K$, $y \in F$. The image of this under τ will be $\tau[k(x), y] = [\tau k(x), \tau(y)] = [k'\tau|_K(x), \tau(y)] = 1$, since $\mathbb{E}'$ is a central extension, so τ induces a $\psi : H \to E$ compatibly with the projections to G, and hence with their kernels. ■

When will G have a *universal* central extension? The answer is: when G is perfect.

Definition: Suppose G is a group, it is *perfect* if $[G, G] = G$, i.e., it is generated by commutators.

Proposition 24 *Every perfect group, G, has a universal central extension.*

Proof: (We can pick up ideas and notation from the previous proof.) As G is perfect, we can restrict $\sigma : H \to G$ to the subgroup $[H, H]$ and still get a surjection. We thus have

$$\begin{tikzcd}
1 \arrow[r] & K \cap [H,H] \arrow[r] \arrow[d] & {[H,H]} \arrow[r, "\sigma"] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & K \arrow[r] & H \arrow[r, "\sigma"'] & G \arrow[r] & 1
\end{tikzcd}$$

It is clear that as the bottom is weakly universal, so is the top one.

We next need a subsidiary result.

Lemma 19 *If $1 \to Ker\,\sigma \to H \xrightarrow{\sigma} G \to 1$ is a weakly universal central extension and H is perfect, then G is perfect and the central extension is universal.*

Proof: The first conclusion should be clear, so we are left to prove 'universal'. Suppose we have $\mathbb{E}'$ as before and obtain two morphism φ and φ', from H to E such that $\sigma'\varphi = \sigma'\varphi' = \sigma$. We have, for $h_1, h_2 \in H$, $\varphi(h_1) = \varphi'(h_1)c$, and $\varphi(h_2) = \varphi'(h_2)d$ for some $c, d, \in L$. we calculate that

$$\varphi(h_1h_2h_1^{-1}h_2^{-1}) = \varphi'(h_1h_2h_1^{-1}h_2^{-1}),$$

since c and d are central in E, but as commutators generate H, $\varphi = \varphi'$ everywhere in H. ■

To complete the proof of the proposition, we show that, back in case $[[H, H]$ is itself perfect. We have

$$[H,H] = \left[\frac{F}{[K,F]}, \frac{F}{[K,F]}\right] = \frac{[F,F]}{[K,F]},$$

now as G is perfect, every element in F can be written in the form $x = ck$ with $c \in [F, F]$ and $k \in K$. (One could say 'F is perfect up to K'.)

Take, now, a $[\overline{x}, \overline{y}] \in [H, H]$, i.e., a commutator of $\overline{x}, \overline{y} \in F/[K, F]$ with $\overline{x}$ denoting the coset $x[K, F]$, etc. Set $x = ck$, $y = d\ell$, $c, d, \in [F, F]$

$$\begin{aligned}
\overline{xyx^{-1}y^{-1}} &= \overline{x}.\overline{y}.\overline{x}^{-1}.\overline{y}^{-1} \\
&= \overline{c}.\overline{d}.\overline{c}^{-1}.\overline{d}^{-1} \\
&= \overline{cdc^{-1}d^{-1}} \in [[H,H],[H,H]]
\end{aligned}$$

since elements of K commute with elements of F mod $[K, F]$. We thus have $[H, H] = [[H, H], [H, H]]$, as claimed. ■

To summarise, suppose we have a group presentation, $G = (X : R)$, of a perfect group, G. This gives us an exact 'presentation sequence'

$$1 \to K \to F \to G \to 1$$

where we abbreviate $N(R)$ to K. There is, then, a short exact sequence:

$$1 \to \frac{K \cap [F,F]}{[K,F]} \to \frac{[F,F]}{[K,F]} \to G \to 1$$

and this is its *universal central extension.*

Remark: The term on the left is the usual formula for the *Schur multiplier* of G and is one of the origins of group *homology*. It gives the Hopf formula for $H_2(G, \mathbb{Z})$, the second homology of G with coefficients in the trivial G-module, $\mathbb{Z}$.

To apply this theory and discussion back to the Steinberg group, $St(R)$, we need to check that $St(R)$ is a perfect group and that the central extension that we have is weakly universal. the first of these is simple.

Lemma 20 *The group $St(R)$ is perfect.*

Proof: We can write any generator $x_{ij}(a)$ as $[x_{ik}(a), x_{kj}(1)]$ for some k other than i or j, so the proof is the same as that $E_n(R)$ is perfect (for $n \geq 3$), that we gave earlier. ■

This leaves us to check that the central extension

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

that we saw earlier is weakly universal (as it will then be universal by the previous lemma).

Suppose that we have

$$1 \to L \to E \xrightarrow{\sigma} E(R) \to 1$$

is a central extension. We have to define a morphism $\psi : St(R) \to E$ projecting down to the identity morphism on $E(R)$. As we have $St(R)$ defined by a presentation, the obvious way to proceed is to find suitable images in E for the generators, $x_{ij}(a)$, and then see if the Steinberg relations are satisfied by them.

To start with, for each generator $x_{ij}(a)$ of $St(R)$, we pick an element, $y_{ij}(a)$, in E such that $\sigma(y_{ij}(a)) = e_{ij}(a)$, the corresponding elementary matrix, which is, of course, the image of $x_{ij}(a)$ in $E(R)$. (Note that any other choice of the $y_{ij}(a)$ will differ from this by a family of elements of the kernel, L, and hence by central elements of E.)

We will prove, or note, various useful identities, which will give us what we need.

- $[u, [v, w]] = [uv, w][w, u][w, v]$ for $u, v, w, \in E$;
- for convenience, for $u \in E$, write $\bar{u} = \sigma(u) \in E(R)$, and for $u, v \in E$, write $u \sim v$ if $uv^{-1} \in L$, then note that if $u \sim u'$ and $v \sim v'$, we have $[u, v] = [u', v']$;
- if $u, v, w, \in E$ with $[\bar{u}, \bar{v}] = [\bar{u}, \bar{w}] = 1$, then

  $$[u, [v, w]] = 1.$$

  To see this, put $a = [u, v]$, $b = [u, w]$, so, by assumption, $\bar{a} = \bar{b} = 1$ and $a, b \in L$. We then have $uvu^{-1} = av$, $uwu^{-1} = bw$, and $[av, bw] = [v, w]$, since $a, b \in L$. Next look at

  $$[u, [v, w]] = u[v, w]u^{-1}[v, w]^{-1} = [uvu^{-1}, uwu^{-1}][v, w]^{-1} = 1$$

  by our previous calculation.

We are now ready to look at the $y_{ij}(a)$s and see how nearly they will satisfy the Steinberg relations, (St1 and St2 of page 99). (They will not necessarily satisfy them 'on-the-nose', but we can use them to get another choice that will work.)

- If $i \neq j$, $k \neq \ell$, so the corresponding ys make sense, and further $i \neq \ell, j \neq k$ (to agree with the condition of the first part of the St2) relation), then $[y_{ij}(a), y_{k\ell}(b)] = 1$. To see this we choose n bigger than all the indices involved here, so that we can have $y_{k\ell}(b) \sim [y_{kn}(b), y_{n\ell}(1)]$, as they give the same element when mapped down to $E(R)$. We thus have

$$[y_{ij}(a), y_{k\ell}(b)] = [y_{ij}(a), [y_{kn}(b), y_{n\ell}(1)]] = 1,$$

  by the above, so the ys do go some way towards what we need, (but the other relations need not hold). We will use them, however, to make a better choice.

- Suppose i, j and n are distinct, and, as always, $a \in R$. Set

$$z^n_{ij}(a) = [y_{in}(a), y_{jn}(1)].$$

  It is easy to see that this depends on i, j and a, and, slightly less obviously, that it does not depend on the choice of the $y_{k\ell}$s. Actually it does not depend on n at all. (The details are left **for you to check**, but use the commutator rules above to show $z^n_{ik}(ab) = [y_{ij}(a), y_{jk}(b)]$. That is independent of n.) We write $z_{ij}(a)$ for $z^n_{ij}(a)$, as n is irrelevant, as long as it is sufficiently large. These $z_{ij}(a)$ will do the trick!

We define $\psi : St(R) \to E$ by defining $\psi(x_{ij}(a)) = z_{ij}(a)$ and will check that $z_{ij}(a)$ satisfies the relations of $St(R)$, (as that will mean that this assignment does define a homomorphism by what is sometimes known as von Dyck's Theorem).

Most have been done (and checking this is again **left to you**), except for

$$z_{ij}(a)z_{ij}(b) = z_{ij}(a + b).$$

Clearly their difference is central in E, but that is not enough. We calculate

$$\begin{aligned}
z_{ij}(a+b) &= z_{ij}(b+a) \\
&= [z_{ik}(b+a), z_{kj}(1)] \quad \text{with} k \neq i, j \\
&= [z_{ik}(b)z_{ik}(a), z_{kj}(1)] \quad \text{as the `difference is central'} \\
&= [z_{ik}(b), z_{ij}(a)]z_{ij}(a)z_{ij}(b) \quad \text{using the first commutator identity above} \\
&= z_{ij}(a)z_{ij}(b)
\end{aligned}$$

as required.

We have checked, in quite a lot of detail, that

Proposition 25

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

is a universal central extension. ∎

4.2.5 Higher algebraic K-theory: some first remarks

Milnor's definition of $K_2(R)$ was initially given in a course at Princeton in 1967. The search for higher algebraic K-groups was then intense; see Weibel's excellent history of algebraic K-theory, [136]. The breakthrough was due to Quillen, who in 1969/70, gave the 'plus construction', which was a method of 'killing' the maximal perfect subgroup of a fundamental group, $\pi_1(X)$. Applying this to the classifying space, $BG\ell(R)$, of the stable general linear group, gave a space $BG\ell(R)^+$, whose homotopy groups had the right sort of properties expected of those mysterious higher groups and so were taken to be $K_n(R) := \pi_n(BG\ell(R)^+)$.

Several other constructions of $K_n(R)$ were given in 1971 and were gradually shown to be equivalent to Quillen's. One of these which was based upon the theory of 'buildings' and upper triangular subgroups was by I. Volodin, [133]. We will look at the general construction in the next few sections as it relates closely to our theme of higher syzygies.

We note that there are several other approaches that were developed at about the same time, but will not be looked at in this chapter. There are also generalisations of these ideas.

4.3 Higher generation by subgroups

We now return to more general discussions relating to presentations, syzygies and rewriting, although we will see the link with ideas and methods from K-theory coming in later on.

Often one has a group, G, and a family $\mathcal{H}$, of subgroups. For example (i) suppose G is given with a presentation, $(X : R)$, then subsets of X yield subgroups of G, and a family of subsets naturally leads to a family of subgroup, or (ii) a group may be a symmetry group of some geometric or combinatorial structure and certain substructures may be fixed by a subgroup, so families of subgroups may correspond to families of substructures. It is common, in this sort of situation, to try to see if information on G can be gleaned from information on the subgroups in $\mathcal{H}$. This will happen to some extent even if it is simply the case that the union of the elements in the subgroups generate G.

A simple example would be if G is generated by three elements a, b and c with some relations (possibly not known or not completely known), $\mathcal{H}$ consists of the subgroup generated by a, and that generated by b. There is a possibility that c is not in the subgroup generated by a and b, but how might this become apparent.

It may be that we have, instead of a presentation of G, presentations of the subgroups in $\mathcal{H}$, can we find a presentation of G, and, more generally, suppose we have knowledge of higher (homotopical or homological) syzygies of the presentations of the subgroups in $\mathcal{H}$, can we find not only a presentation of G, but build up knowledge of (at least some of) the syzygies for that presentation?

The key to attacking these problems is a knowledge of the way that the subgroups interact and by building up knowledge of the correspondence between the combinatorics of that interaction and of the induction process of building out from $\mathcal{H}$ to the whole group, G.

Various instances of this process had been studied, notably by Tits, e.g. in [124–126], since, in the situations studied in those papers, the combinatorics leads to the building of a Tits system. They also occur in the work of Behr, [17] and Soulé, [120], but, because of their general approach and the explicit link made to identities among relations, we will use the beautiful paper by Abels and Holz, [1]. This, and some subsequent developments, provides the basis for a way of calculating some syzygies in some interesting situations.

There is also a strong link with Volodin's approach to higher algebraic K-theory, but that will be slightly later in the notes. Here we sketch some of the background and intuition, giving some very elementary examples. When we have more knowledge of how to work with syzygies using both homotopical and homological methods, whether 'crossed' or not, we will return to look in more detail. We will see that this study of 'higher generation' leads in some interesting directions, towards geometric constructions and concepts of use elsewhere.

4.3.1 The nerve of a family of subgroups

We start, therefore, with a group, G, and a family, $\mathcal{H} = \{H_i \mid i \in I\}$ of subgroups of G. Each subgroup, H, determines a family of right cosets, H_g, which cover the *set*, G. Of course, these partition G, so there are no non-trivial intersections between them. If we use *all* the right cosets, H_ig, for *all* the H_i in $\mathcal{H}$, then, of course, we expect to get non-trivial intersections.

Remark: There is some disagreement as to which terminology for cosets is the most logical, so we should say exactly what we mean by 'right coset'. A subgroup H of G give a left action, $H \curvearrowright G$ on the *set*, G, by multiplication on the left, and hence a groupoid whose connected components are the *right* cosets, Hg. The terminology 'right coset' corresponds to the g being on the right. If we considered the right action then we would have left cosets in the corresponding role.

Another notational point is that when writing cosets, we follow the usual rule that there is some informal set of coset representatives being used, or more exactly that the notation looks like that! This can be delicate if we step outside a set based situation, as choosing a set of coset representatives uses the axiom of choice, and in some contexts that would be 'dodgy'.

Let

$$\mathfrak{H} = \coprod_{i \in I} H_i \backslash G = \{H_ig \mid H_i \in \mathcal{H}\},$$

where the g is more as an indicator of right cosets than strictly speaking an index. This is the family of all right cosets of subgroups in $\mathcal{H}$. This covers G and we write $N(\mathfrak{H})$ for the corresponding simplicial complex, which is the *nerve* of this covering.

In many situations, 'nerves' in some form are used to help 'integrate' *local* information into *global*, since they record the way the 'localities' of the information fit together. (We will refer to this type of problem as a 'local-to-global' problem. They occur in many different contexts.) We have met nerves of categories, and will later meet nerves of open covers of topological spaces, but in that latter situation, the topological features of the construction are not central to that construction. We will consider the fairly general case of the nerve of a relation in a while, but for the moment, we will give a working definition, specific to the application that we have in mind here. We will refine and extend that definition later on.

Definition: Let G be a group and $\mathcal{H}$ a family of subgroups of G. Let $\mathfrak{H}$ denote the corresponding covering family of right cosets, H_ig, $H_i \in \mathcal{H}$. (We will write $\mathfrak{H} = \mathfrak{H}(G, \mathcal{H})$ or even $\mathfrak{H} = (G, \mathcal{H})$, as a shorthand as well.) The *nerve* of $\mathfrak{H}$ is the simplicial complex, $N(\mathfrak{H})$, whose vertices are the cosets, H_ig, $i \in I$, and where a non-empty finite family, $\{H_ig_i\}_{i\in J}$, is a simplex if it has non-empty intersection.

Examples: (i) If $\mathcal{H}$ consists just of one subgroup, H, then $\mathfrak{H}$ is just the set of cosets, $H\backslash G$ and $N(\mathfrak{H})$ is 0-dimensional, consisting just of 0-simplices / vertices.

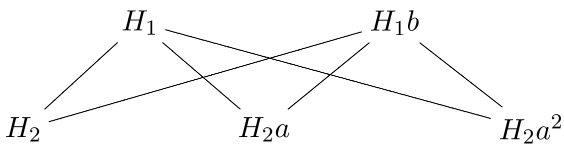
(ii) If $\mathcal{H} = \{H_1, H_2\}$, (and H_1 and H_2 are not equal!), then any right H_1 coset, H_1g, will intersect some of the right H_2-cosets, for instance, $H_1g \cap H_2g$ always contains g. The nerve, $N(\mathfrak{H})$, is a bipartite graph, considered as a simplicial complex. (If the group G is finite, or more generally, if both subgroups have finite index, the number of edges will depend on the sizes or indcices of H_1,

H_2 and $H_1 \cap H_2$.) It is just a graphical way of illustrating the intersections of the cosets, a sort of intersection diagram. (There is an error in [10] in which it is claimed that each coset H_1 will intersect with each of those of H_2.)

As a specific very simple example, consider:

- $S_3 \equiv (a, b : a^3 = b^2 = (ab)^2 = 1)$, (so a denotes, say, the 3-cycle $(1\ 2\ 3)$ and b, a transposition $(1\ 2)$).
- Take $H_1 = \langle a \rangle = \{1, (1\ 2\ 3), (1\ 3\ 2)\}$, yielding two cosets H_1 and $H_1 b$.
- Similarly take $H_2 = \langle b \rangle = \{1, (1\ 2)\}$ giving cosets H_2, $H_2 a$ and $H_2 a^2$.

The covering of S_3 is then $\mathfrak{H} = \{H_1, H_1 b, H_2, H_2 a, H_2 a^2\}$ and has nerve



4.3.2 n-generating families

Abels and Holz, [1], give the following definition:

Definition: A family, $\mathcal{H}$, of subgroups of G is called *n-generating* if the nerve, $N(\mathfrak{H})$, of the corresponding coset covering is $(n-1)$-connected, i.e., $\pi_i N(\mathfrak{H}) = 0$ for $i < n$.

The following results illustrate the idea and motivate the terminology. (They are to be found in [1].)

Proposition 26 *The group, G, is generated by the union of the subgroups, H, in $\mathcal{H}$ if, and only if, $N(\mathfrak{H})$ is connected.*

We will take this apart rather than use the short proof given in [1]. (Hopefully this will show how the idea works and how simple minded the proof can be!)

Proof: Suppose we have that G is generated by the various H in $\mathcal{H}$ and we are given two vertices Hg_1 and Kg_2 for $H, K \in \mathcal{H}$. (The case $H = K$ is allowed here.) Of course, $g_1 g_2^{-1} \in G$, so is a product of elements from the various H_is, say, $g_1 g_2^{-1} = h_{i_1} \ldots h_{i_n}$ with $h_{i_k} \in H_{i_k}$. (This observation suggests an induction on the length of this expression.)

To 'test the water', we assume $g_1 g_2^{-1} = h_1 \in H_1$, but then $g_1 \in Hg_1 \cap H_1 g_2$ and also $g_2 \in H_1 g_2 \cap K g_2$. (We can indicate this diagrammatically as

$$\begin{tikzcd} Hg_1 \arrow[r, dash, "g_1"] & H_1 g_2 \arrow[r, dash, "g_2"] & Kg_2, \end{tikzcd}$$

where each edge is decorated by an element that *witnesses* that the intersection of the two cosets is non-empty.)

If we try next with $g_1 g_2^{-1} = h_1 h_2$, then $g_1 = h_1 h_2 g_2$, so we have

$$\begin{tikzcd} Hg_1 \arrow[r, dash, "g_1"] & H_1(h_2 g_2) \arrow[r, dash, "h_2 g_2"] & H_2 g_2 \arrow[r, dash, "g_2"] & Kg_2, \end{tikzcd}$$

and the pattern gives the model for an induction on the length of the expression giving $g_1g_2^{-1}$ in terms of elements of the H_is. (Note the link between the expression and the path is very simple.)

Conversely, suppose that $N(\mathfrak{H})$ is connected, then if $g \in G$, we look at Hg and H for some choice of H. There is a sequence of edges in $N(\mathfrak{H})$ joining these two vertices. We examine the length, ℓ, of such an edge path. If $\ell = 1$, there is some $h \in H \cap Hg$, so $g \in H$. If $\ell = 2$,

$$H \overset{x_1}{\longrightarrow} H'g_1 \overset{x_2}{\longrightarrow} Hg,$$

and we have $x_1 = h_1 = h_2g_1$ with $h_2 \in H'$, whilst $x_2 = h_3g_1 = h_4g$. We thus obtain $g = h_4^{-1}h_3g_1$ and $g_1 = h_2^{-1}h_1$, so $g = h_4^{-1}h_3h_2^{-1}h_1$, i.e., we have an expansion of g in terms of elements of the various Hs. A proof of the general case is now easy. ■

We next form a diagram, $\mathcal{D}$, consisting of the subgroups, H_i, and all their pairwise intersections, together with the natural inclusions, and we write $H := \underset{\cap}{\sqcup}\mathcal{H}$ for *colim* $\mathcal{D}$. (Note that this colimit is within the category of groups.) More exactly, there is a poset $\{H_j, H_j \cap H_k \mid j, k \in I\}$, ordered by inclusion and $\mathcal{D}$ is the inclusion of this diagram into the category of groups. There is a presentation of H with generators x_g, $g \in \bigcup H_j$ and with relations $x_g \cdot x_h = x_{gh}$ if g and h are both in some H_i. (This group, H, is thus a 'coproduct' with amalgamated subgroups.)

There is an obvious homomorphism

$$H = \underset{\cap}{\sqcup}\mathcal{H} \to G$$

induced by the inclusions.

Proposition 27 *The family, $\mathcal{H}$, is 2-generating if, and only if, the natural homomorphism,*

$$H = \underset{\cap}{\sqcup}\mathcal{H} \to G,$$

is an isomorphism. ■

In fact,

Proposition 28 *There are isomorphisms:*
(a) $\pi_0 N(\mathfrak{H}) \cong G/\langle \bigcup H_j\rangle$;
(b) $\pi_1 N(\mathfrak{H}) \cong Ker(\underset{\cap}{\sqcup}\mathcal{H} \to G)$. ■

We almost have shown (a) in our above argument, but will postpone more detailed proofs until later. (They are, in fact, quite easy to give by direct calculation.)

Remark: It is often helpful to take the family, $\mathcal{H}$, of subgroups and to close it up under (finite) intersection and sometimes the inclusion order on the intersections comes in useful as well. This closure operation does not change the homotopy type of the nerve of the corresponding coverings by cosets, in fact, the process of taking intersections corresponds to taking the barycentric subdivision of the original nerve.

4.3.3 A more complex family of examples

An important example of the above situation is in algebraic K-theory. It occurs with the general linear group, $G\ell_n(R)$, of invertible $n\times n$ matrices together with a family of subgroups corresponding to lower triangular matrices, ... but with some subtleties involved.

Let R be an associative ring with identity and n a positive integer.

Let $\Delta = \{(i,j) \mid i \neq j, 1 \leq i,j \leq n\}$ be the set of non-diagonal positions in an $n \times n$ array. We will say that a subset, $\alpha \subseteq \Delta$, is *closed* if

$$(i,j) \in \alpha \text{ and } (j,k) \in \alpha \text{ implies } (i,k) \in \alpha.$$

Note that if $(i,j) \in \alpha$ and α is closed then $(j,i) \notin \alpha$.

Let $\Phi = \{\alpha \subseteq \Delta \mid \alpha \text{ is closed}\}$. There is a reflexive relation $\leq$ on Φ by $\alpha \leq \beta$ if $\alpha \subseteq \beta$. These αs are transitive relations on subsets of the set of integers from 1 to n, so essentially order the elements of the subset. The reason for their use is the following: suppose $(i,j) \in \Delta$ and $r \in R$. The *elementary matrix*, $\varepsilon_{ij}(r)$, is the matrix obtained from the identity $n \times n$ matrix by putting the element r in position (i,j),

$$\text{i.e.,} \qquad \varepsilon_{ij}(r)_{k,l} = \begin{cases} 1 & \text{if } k = l \\ r & \text{if } (k,l) = (i,j) \\ 0 & \text{otherwise .} \end{cases}$$

Let $G\ell_n(R)_\alpha$, for $\alpha \in \Phi$, denote the subgroup of $G\ell_n(R)$ generated by

$$\{\varepsilon_{ij}(r) \mid (i,j) \in \alpha, r \in R\}.$$

It is easy to see that $(a_{kl}) \in G\ell_n(R)_\alpha$ if and only if

$$a_{k,l} = \begin{cases} 1 & \text{if } k = l \\ \text{arbitrary} & \text{if } (i,j) \in \alpha \\ 0 & \text{if } (i,j) \in \Delta \backslash \alpha. \end{cases}$$

If $\alpha \leq \beta$, then there is an inclusion, $G\ell_n(R)_{\alpha\leq\beta}$ of $G\ell_n(R)_\alpha$ into $G\ell_n(R)_\beta$.

We will consider the $G\ell_n(R)_\alpha$ as forming a family, $\mathcal{G}\ell_n(R)$, of subgroups of $G\ell_n(R)$.

Remark: Although a similar idea is found in Wagoner's paper [134], I actually learnt the idea for this approach to these subgroups from papers by A. K. Bak, [8, 9], and, with others, in [10], and from talks he gave in Bangor and Bielefeld. In these sources, this construction leads on to a discussion of his notion of a global action, and, in the third paper cited, the variant known as a *groupoid atlas*. The motivation, there, is to study the *unstable* algebraic K-theory groups, whilst Volodin's original and Wagoner's approach are more centred on the stable version.

There is a lot more that could be said about these groupoid atlasses, which were introduced to handle the intrinsic homotopy involved in Volodin's definition of a form of algebraic K-theory, [133]. We will not use them explicitly here, but will attempt to show the link between the above and the question of syzygies, higher generation by subgroups, etc.

The nerve of this family would consist of the cosets of these subgroups, linked via their intersections. We need to extract another description of the homotopy type of this simplicial complex and for that will examine the intersections of cosets, and of the subgroups. We will do this in a slightly strange way in as much as we will turn first, or rather after some preparation, to descriptions related to Volodin's version of the higher K-theory of an associative ring. Our approach will be via *Volodin spaces* as used, for instance, in a paper by Suslin and Wodzicki, [122] and then an examination of the various nerves of a relation, before returning to this setting.

4.3.4 Volodin spaces

Let X be a non-empty set, and denote by $E(X)$, the simplicial set having $E(X)_p = X^{p+1}$, so a p-simplex is a $p+1$ tuple, $\underline{x} = (x_0, \ldots, x_p)$, each $x_i \in X$, and in which

$$d_i(\underline{x}) = (x_0, \ldots, \hat{x_i}, \ldots x_p),$$

and

$$s_j(\underline{x}) = (x_0, \ldots, x_j, x_j, \ldots x_p),$$

so d_i omits x_i, whilst s_j repeats x_j.

Lemma 21 *The simplicial set, $E(X)$, is contractible.*

Proof: We thus have to prove that the unique map $E(X) \to \Delta[0]$ is a homotopy equivalence. (That this is the case is well known, but we will none the less give a sketch proof of it as firstly we have not assumed that much knowledge of simplicial homotopy and also as it gives some interesting insights into that subject in a very easy situation.) We pick some $a_0 \in X$ and obtain a map $\Delta[0] \xrightarrow{a_0} E(X)$ by mapping the single 0-simplex of $\Delta[0]$ to the 0-simplex, (a_0) in $E(X)$. We now show that the identity map on $E(X)$ is homotopic to the composite map, $E(X) \to \Delta[0] \xrightarrow{a_0} E(X)$, that 'sends all simplices to a_0'.

We will look at simplicial homotopies in more detail later, (in particular around page **??**), but clearly, a homotopy $h : f \simeq g : K \to L$, between two simplicial mapsa $f, g : K \to L$, should be a simplicial map $h : K \times \Delta[1] \to L$, restricting to f and g on the two ends of $K \times \Delta[1]$.. Here we need a homotopy $h : E(X) \times \Delta[1] \to E(X)$ and we look at what this must be on a cylinder over a simplex, $(x_0, \ldots, x_p)$. To see what to do, look at almost the simplest case, $p = 1$, then a schematic representation of h on $(x_0, x_1) \times \Delta[1]$ must look like:

$$\begin{tikzcd} a_0 \arrow[r] & a_0 \\ x_0 \arrow[u] \arrow[r] \arrow[ur] & x_1 \arrow[u] \end{tikzcd}$$

More precisely, the two simplices of $E(X) \times \Delta[1]$ that we need have two forms

$$\sigma_1 = ((x_0, 0), (x_1, 0), (x_1, 1))$$

and

$$\sigma_2 = (x_0, 0), (x_0, 1), x_1, 1))$$

being, respectively the bottom right and the top left hand ones. We need $h(\sigma_1) = (x_0 x_1, a_0)$ and $h(\sigma_2) = (x_0, a_0, a_0)$. Now it is easy to see how to set up h, in general, giving the required contracting homotopy. ■

Remark: Any homotopy can be specified by a family of maps, $h^n_i : K_n \to L_{n+1}$, satisfying some rules that will be given later (page **??**). It is then easy to specify the $h^n_i : E(X)_n \to E(X)_{n+1}$ generalising the formula we have given above. (We **leave this to you** if you have not seen it before, as it is easy, but also instructive.)

The case we are really interested in is when we replace the general set, X, by the underlying set of a group, G. (As usual, we will not introduce a special notation for the underlying set of G, just writing G for it.) In this case we have the simplicial set $E(G)$ and the group, G, acts freely on $E(G)$ by

$$g \cdot (g_0, \ldots, g_p) = (gg_0, \ldots, gg_p).$$

(Here we have used a left action of G, and **leave you to check** that the evident right action could equally well be used.) The quotient simplicial set of orbits, will be denoted $G\backslash E(G)$. It is often useful to write $[g_1, \ldots, g_p]$ for the orbit of the p-simplex $(1, g_1, g_1g_2, \ldots, g_1g_2 \ldots g_p) \in E(G)_p$.

It is 'instructive' to calculate the faces and degeneracy maps in this notation. We will only look at $[g_1, g_2]$ in detail. This element has representative $(1, g_1, g_1g_2)$. We thus have:

- $d_0(1, g_1, g_1g_2) = (g_1, g_1g_2) \equiv (1, g_2)$, so $d_0[g_1, g_2] = [g_2]$;
- $d_1(1, g_1, g_1g_2) = (1, g_1g_2)$, so $d_1[g_1, g_2] = [g_1g_2]$;
- $d_2(1, g_1, g_1g_2) = (1, g_1)$, so $d_2[g_1, g_2] = [g_1]$.

(That looks familiar!)

For the degeneracies,

- $s_0(1, g_1, g_1g_2) = (1, 1, g_1, g_1g_2)$, so $s_0[g_1, g_2] = [1, g_1, g_2]$;
- $s_1(1, g_1, g_1g_2) = (1, g_1, g_1, g_1g_2)$, so $s_1[g_1, g_2] = [g_1, 1, g_2]$;

and similarly $s_2[g_1, g_2] = [g_1, g_2, 1]$.

The general formulae are now easy to guess and to prove - so they will be **left to you**, and then the following should be obvious.

Lemma 22 *There is a natural simplicial isomorphism,*

$$G\backslash E(G) \xrightarrow{\cong} Ner(G[1]) = BG.$$

■

We thus have that $G\backslash E(G)$ is a 'classifying space' for G.

We note that this shows that $G\backslash E(G)$ is a Kan complex, since we already have that $Ner(G[1])$ is one. It is easy enough to check it directly. Of course, $E(G)$ is Kan as well. Jumping ahead of ourselves, we will sketch that the fundamental group of $G\backslash E(G)$ is $\pi_1(G\backslash E(G)) \cong G$, whilst for $k > 1$, $\pi_k(G\backslash E(G))$ is trivial. (We will have to 'fudge' the details as they either need material that will not be directly handled in these notes (and hence, for which the reader is referred to standard texts on simplicial homotopy theory), or they may depend on ideas that will be only explored later on in the notes, so we will sketch enough to whet the appetite!)

First we take on trust that if K is a connected Kan complex, then the k^{th} homotopy group of K can be 'calculated' by looking at homotopy classes of mappings from the boundary of a $k+1$-simplex into K, based at a base point. If you have a map, $\partial\Delta[k+1] \to Ner(G[1])$, then you have *all* the information needed to extend it to a map defined on $\Delta[k+1]$, i.e., the map you started with is null homotopic. (If you want more intuition on this, try looking at the case $k = 2$ and writing down what the various faces in $\partial\Delta[3]$ will give and then see how they determine a 3-simplex in $Ner(G[1])$.)

For dimension 1, the construction of π_1 is, of course, that of the fundamental group(oid), so gives a presentation with set of generators $\{[g] \mid g \in G\}$ and, for each pair (g_1, g_2), a relation r_{g_1,g_2} corresponding to $[g_1, g_2] \in G\backslash E(G)_2$, and which gives $[g_1][g_2][g_1g_2]^{-1}$, but this was our prime example of a presentation of G, so $\pi_1(G\backslash E(G)) \cong G$.

There is, here, another **useful fact for the reader to check**. The quotient map from $E(G)$ to $G\backslash E(G)$ is a Kan fibration (and this is a **useful example to do in detail** if you are not that conversant with Kan fibrations). The fibre of this quotient map is a constant (or 'discrete') simplicial set with *value* G, so is a $K(G, 0)$. As is well known, and as we will introduce and use later, there is a long exact sequence of homotopy groups for any pointed fibration sequence, $F \to E \to B$, so we can apply this to

$$K(G, 0) \to E(G) \to G\backslash E(G)$$

to get $\pi_i(G\backslash E(G) \cong \pi_{i-1}(K(G, 0))$ and another proof that $G\backslash E(G)$ is an 'Eilenberg Mac Lane space' for G, i.e., a $K(G, 1)$ in the usual notation, (... and yes, this is related to covering spaces ...).

Returning to the construction of what are called 'Volodin spaces' (cf. [122]), we put ourselves back in the context of a group, G, and a family, $\mathcal{H}$, of subgroups of G. We suppose that $\mathcal{H} = \{H_i \mid i \in I\}$ for some indexing set, I. (We may assume extra structure on I, as before, when we get further into the construction.)

Definition: (Suslin-Wodzicki, [122], p. 65.) We denote by $V(G, \mathcal{H})$, or $V(\mathfrak{H})$, the simplicial subset of $E(G)$ formed by simplices, $(g_0, \ldots, g_p)$, that satisfy the condition that there is some $i \in I$ such that, for all $0 \leq j, k \leq p$, $g_jg_k^{-1} \in H_i$.

The simplicial set, $V(G, \mathcal{H})$, will be called the *Volodin space* of $(G, \mathcal{H})$.

Remark: The actual definition given in [122] uses $g_j^{-1}g_k \in H_i$, as there the convention on cosets is gH rather than our Hg.

The subobject, $V(G, \mathcal{H})$, of $E(G)$ is a G-subobject, i.e., it is invariant under the action of G. The corresponding quotient simplicial set $G\backslash V(G, \mathcal{H})$ coincides with the union of the BH_i within the classifying space, BG.

Remark: The construction of $V(G, \mathcal{H})$ is usually ascribed to Volodin in his approach to the higher K-theory groups of a ring, but in fact, the basic construction is essentially much older, being due to Vietoris in the 1920s, but in a different setting, namely that of a simplicial complex associated to an open covering of a space. This was further studied by Dowker, [55], in 1952, where he abstracted the situation to construct two simplicial complexes from a relation between two sets.

4.3.5 The two nerves of a relation: Dowker's construction

The results of the next few sections are of much more general use than just for a group and a family of its subgroups. We therefore present things in an abstract version.

Let X, Y be sets and R a relation between X and Y, so $R \subseteqq X \times Y$. We write xRy for $(x, y) \in R$.

Fairly generic example: Let X be a set (often a topological space) and Y be a collection of (usually open) subsets of X covering X, i.e., $\bigcup Y = X$. The classical case is when Y is an index set for an open cover of X. The relation is xRy if and only if $x \in y$, or, more exactly, x is in the subset indexed by y.

Returning to the abstract setting, we define two simplicial complexes associated to R, as follows:

(i) $K = K_R$:

(a) the set of vertices is the set X;

(b) a p-simplex of K is a set $\{x_0, \cdots, x_p\} \subseteq X$ such that there is some $y \in Y$ with $x_i Ry$ for $i = 0, 1, \ \cdots, \ p$.

(ii) $L = L_R$:

(a) the set of vertices is the set, Y;

(b) p-simplex of K is a set $\{y_0, \cdots, y_p\} \subseteq Y$ such that there is some $x \in X$ with xRy_j for $j = 0, 1, \ \cdots, \ p$.

Clearly the two constructions are in some sense dual to each other. The original motivating example was as above. It had X, a space, and $Y = \mathcal{U} = \{U_\alpha : \alpha \in A\}$, an open cover of X, and, in that case, K_R is the *Vietoris complex* of $\mathcal{U}$, $V(\mathcal{U})$ or $V(X,\mathcal{U})$, of the cover. The 'dual' construction has the open cover, $\mathcal{U}$, or better, the indexing set, A, as its set of vertices, and $\sigma = \langle \alpha_0, \alpha_1, ..., \alpha_p \rangle$, belongs to L_R if and only if the open sets, U_{α_j}, $j = 0, 1, \ldots, p$, have non-empty common intersection. This is the simplicial complex known as the *Čech complex*, *Čech nerve* or simply, *nerve*, of the open covering, $\mathcal{U}$, and it will be denoted $N(X,\mathcal{U})$, or $N(\mathcal{U})$. We will have occasion to repeat this definition later, both when considering Čech non-Abelian cohomology, (starting on page **??**), and also when looking at triangulations when examining methods of constructing some simple topological quantum field theories, page **??**.

We will extend the terminology so that for a given relation, R, K_R will be called the *Vietoris nerve* of R, whilst L_R is its *Čech nerve.* (This is rather arbitrary as the Vietoris nerve of R is the Čech nerve of the opposite relation, R^{op}, from Y to X.)

In the situation in this chapter, we have a pair, $(G, \mathcal{H})$, and X is G, whilst Y is the family, $\mathfrak{H}$, of right cosets of subgroups from the family $\mathcal{H}$. The relation is 'xRy if and only if $x \in y$'.

The simplicial complex, K_R, thus has G as its set of vertices and $(g_0, \ldots, g_p)$ is a p-simplex of K_R if, and only if, all the g_ks are in some common right coset, $H_i x$, in the family $\mathfrak{H}$. It is then just a routine calculation to check that this is the same as saying that the simplex is in $V(\mathfrak{H})$. In other words, the Volodin complex of $(G, \mathcal{H})$ is the same as the Vietoris complex of $\mathfrak{H}$, and it is convenient that both names begin with the letter 'V'! The one difference is that the Vietoris complex is a simplicial complex, whilst the Volodin space is a simplicial set. For each p-simplex $\{g_0, \ldots, g_p\}$, of $V(\mathfrak{H})$, there are $p!$ simplices in the Volodin space.

The corresponding Čech nerve, L_R, is $N(\mathfrak{H})$ as introduced earlier, so if $\sigma \in N(\mathfrak{H})_p$ $\sigma = \{H_0g_0, \cdots, H_pg_p\}$ with the requirement that $\cap\sigma = \underset{i=0}{\cap}{}^p H_ig_i \neq \emptyset$.

Before turning to Dowker's result, we will examine barycentric subdivisions as these play a neat role in his proof.

4.3.6 Barycentric subdivisions

Combinatorially, if K is a simplicial complex with vertex set, V_K, then one associates to K the partially ordered set of its simplices. (We avoid our earlier notation of $V(K)$ for the vertex set as being too ambiguous here.) Explicitly we write $S(K)$ for the set of simplices of K and $(S(K), \subseteq)$ for the partially ordered set with $\subseteq$ being the obvious inclusion. The *barycentric subdivision*, K', of K has $S(K)$ as its set of vertices and a finite set of vertices of K' (i.e., simplices of K) is a simplex of K' if it can be totally ordered by inclusion.) We may sometimes write $Sd(K)$ instead of K'.)

Remark: It is important to note that there is, in general, no natural simplicial map from K' to K. If, however, V_K is given an order in such a way that the vertices of any simplex in K are totally ordered (for instance by picking a total order on V_K), then one can easily specify a map,

$$\varphi : K' \to K,$$

by:
if $\sigma' = \{x_0, \cdots, x_p\}$ is a vertex of K' (so $\sigma' \in S(K)$), let $\varphi\sigma'$ be the least vertex of σ' in the given fixed order.

This preserves simplices, but reverses order so if $\sigma_1' \subset \sigma_2'$ then $\varphi(\sigma_1') \geq \varphi(\sigma_2')$.

If one changes the order, then the resulting map is *contiguous*:

Definition: Let $\varphi, \psi : K \to L$ be two simplicial maps between simplicial complexes. They are said to be *contiguous* if for any simplex σ of K, $\varphi(\sigma) \cup \psi(\sigma)$ forms a simplex in L.

Contiguity gives a constructive form of homotopy applicable to simplicial maps between simplicial complexes.

If $\psi : K \to L$ is a simplicial map, then it induces $\psi' : K' \to L'$ after subdivision. As there is no way of knowing/picking compatible orders on V_K and V_L in advance, we get that on constructing

$$\varphi_K : K' \to K$$

and

$$\varphi_L : L' \to L$$

that $\varphi_L\psi'$ and $\psi\varphi$ will be contiguous to each other, but rarely equal.

4.3.7 Dowker's lemma

Returning to K_R and L_R, we order the elements of X and Y, then suppose y' is a vertex of L'_R, so $y' = \{y_0, \cdots, y_p\}$, a simplex of L_R and there is an element $x \in X$ with $xRy_i, i = 0, 1, \cdots, p$. Set $\psi y' = x$ for one such x.

If $\sigma = \{y'_0, \cdots, y'_q\}$ is a q-simplex of L'_R, assume y'_0 is its least vertex (in the inclusion ordering)

$$\varphi_L(y'_0) \in y'_0 \subset y' \text{ for each } y_i \in \sigma,$$

hence $\psi y'_i R \varphi_L(y'_0)$ and the elements $\psi y'_0, \cdots, \psi y'_q$ form a simplex in K_R, so $\psi : L'_R \to K_R$ is a simplicial map. It, of course, depends on the ordering used and on the choice of x, but any other choice $\bar{x}$ for $\psi y'$ gives a contiguous map.

Reversing the rôles of X and Y in the above, we get a simplicial map,

$$\bar{\psi} : K'_R \to L_R.$$

Applying barycentric subdivisions again gives

$$\bar{\psi}' : K''_R \to L'_R,$$

and composing with $\psi : L'_R \to K_R$ gives a map

$$\psi\bar{\psi}' : K''_R \to K_R.$$

Of course, there is also a map

$$\varphi_K \varphi'_K : K''_R \to K_R.$$

Proposition 29 *(Dowker, [55] p.88). The two maps $\varphi_K \varphi'_K$ and $\psi\bar{\psi}'$ are contiguous.*

Before proving this, note that contiguity implies homotopy and that $\varphi\varphi'$ is homotopic to the identity map on K_R after realisation, i.e., this shows that

Corollary 5

$$|K_R| \simeq |L_R|.$$

■

The actual homotopy depends on the ordering of the vertices and so is not natural.

Proof of the Proposition:

Let $\sigma''' = \{x''_0, x''_1, \cdots, x''_q\}$ be a simplex of K''_R and as usual assume x''_0 is its least vertex, then for all $i > 0$

$$x''_0 \subset x''_i.$$

We have that φ'_K is clearly order reversing, so $\varphi'_K x''_i \subseteq \varphi'_K x''_0$. Let $y = \bar{\varphi}\varphi'_K x''_0$, then for each $x \in \varphi'_K x''_0$, xRy. Since $\varphi_K \varphi'_K x''_i \in \varphi'_K x''_i \subseteq \varphi'_K x''_0$, we have $\varphi_K \varphi'_K x''_i Ry$.

For each vertex x' of x_i'', $\bar{\psi}x' \in \bar{\psi}'x_i''$, hence as $\varphi_K' x_0'' \in x_0'' \subset x_i''$, $y = \bar{\psi}\varphi_K' x x_0'' \in \bar{\psi}' x_i''$ for each x_i'', so for each x_i'', $\psi\bar{\psi}' x_i'' R y$, however we therefore have

$$\varphi_k \varphi_K'(\sigma'') \cup \psi\bar{\psi}(\sigma''') = \bigcup \varphi_k \varphi_K'(x_i'') \cup \psi\bar{\psi}; x_i''$$

forms a simplex in K_R, i.e., $\varphi_K \varphi_K'$ and $\psi\bar{\psi}'$ are contiguous. ■

To prove this we had to choose orders on the two sets, and thus we were working with the non-degenerate simplices of the corresponding simplicial sets. (Abels and Holz, [1], use the neat notation of writing $N^{simp}(R)$, etc. for the corresponding simplicial set, either dependent on order or taking all possible orders, i.e., a p-tuple is a simplex in the simplicial set if its underlying set of elements is a simplex in the simplicial complex. Which method is used make essentially no difference most of the time. Their notation can be useful, but we will tend to ignore the difference as the homotopy groups and homotopy types are independent of which approach one takes.)

4.3.8 Flag complexes

The construction of the barycentric subdivision is closely related to that of a flag complex of a poset.

Suppose that $\mathcal{P} = (P, \leq)$ is a partially ordered set (poset), then we can consider is as a category and hence look at its nerve. This is the associated simplicial set of the flag complex of $\mathcal{P}$, which is a simplicial complex, whose construction uses some ideas that can be of use later on, so we will briefly discuss how it relates to our situation.

Definition: A subset, σ,of $\mathcal{P} = (P, \leq)$ is said to be a *flag* if it satisfies, for all $x.y \in P$, either $x \leq y$ or $y \leq x$.

A finite non-empty flag, thus, is a linearly ordered subset of P, i.e., is of the form $\{x_0, \ldots x_p\}$, where $x_0 < \ldots x_n$ are elements of the set P.

Definition: Let $\mathcal{P} = (P, \leq)$ be a poset. The *flag complex*, $Flag(\mathcal{P})$ of $\mathcal{P}$ is the simplicial complex having the elements of P as its vertices and in which a p-simplex will be a non-empty flag, $x_0 < \ldots x_n$. in $\mathcal{P}$.

This is often also called the *order complex* of the poset.

Lemma 23 *The flag complex construction gives a functor*

$$Flag : Posets \to SimComp,$$

from the category of partially ordered sets and order preserving maps, to the category of simplicial complexes and simplicial morphisms between them. ■

As a simplicial complex, K, consists of a set, $V(K)$ of vertices and a set $S(K) \subseteq P(V(K)) - \{\emptyset\}$, $S(K)$ can naturally be ordered by inclusion to get a partially ordered set $U(K) = (S(K), \subseteq)$. This gives a functor,

$$U : SimpComp \to Posets.$$

The composite functor,

$$Flag \circ U : SimpComp \to SimpComp$$

is the *barycentric subdivision* functor, Sd.

If X is a set and $\mathcal{U} = \{U_i \mid i \in I\}$ is a family of subsets of X, we may think of $\mathcal{U}$ as being ordered by inclusion and thus get a poset. (Of course, this will only be significant if there are some inclusions between the U_is, for instance if $\mathcal{U}$ is closed under finite intersection.) This gives a poset, $(\mathcal{U}, \subseteq)$ and we will abbreviate $Flag(\mathcal{U}, \subseteq)$ to $F(\mathcal{U})$.

The links between nerves and flag complexes are strong.

Proposition 30 *(Abels and Holz, [1], p. 312) Suppose given $(X,\mathcal{U})$ as above, and that $\mathcal{U}$ is such that, if U and V are in $\mathcal{U}$ and $U \cap V$ is not empty, then $U \cap V \in \mathcal{U}$, then there is a natural homotopy equivalence,*

$$|N(\mathcal{U})| \simeq |F(\mathcal{U})|.$$

We cannot give a full proof here as it involves a result, namely Quillen's Theorem A, [112], that will not be discussed in these notes. We can however give a sketch (based on the treatment in [1]).

Sketch proof: Abusing notation so as to consider the simplicial complex, $N(\mathcal{U})$, as being the same as the poset of its simplices, we define a mapping:

$$f : N(\mathcal{U}) \to \mathcal{U}$$

sending $\sigma = \{U_0, \ldots, U_p\}$ to $U_\sigma = \cap_{i=0}^{p} U_i$. This is order reversing. (Note that it, of course, needs $\mathcal{U}$ to be closed under pairwise non-empty intersections.) Writing $\mathcal{U}^{op}$ for the poset, $(\mathcal{U}, \supseteq)$, that is with the opposite order, the poset $U \downarrow f$ of objects under some $U \in \mathcal{U}^{op}$ is just $\{\tau \in N(\mathcal{U}) \mid U_\tau \supseteq U\}$, so is a directed poset, and hence is contractible. By Quillen's theorem A, f induces a homotopy equivalence as claimed. ∎

Remark: An interesting variant of these nerve and flag complex constructions combines some aspects of the Vietoris complex construction with the idea of flags to construct a bisimplicial set. A (p, q)-simplex will be pair consisting of a subset $\{x_0, \ldots, x_p\}$ of X together with a flag $U_0 \subset U_1 \subset \ldots \subset U_q$, such that all the x_i are in U_0. We will not explore this idea here as we have not discussed bisimplicial sets in any detail yet.

Within geometric group theory, the term 'flag complex' is also applied to a closely related, but distinct, concept. These 'flag complexes' are abstract simplicial complexes that satisfy a particular defining property, rather than being defined by how they are constructed. We will see other similar ideas later on in less geometric contexts, but for the moment will give a brief discussion based on the treatment of Bridson and Haefliger, [23], p. 210.

Definition: Let L be a simplicial complex with set of vertices $V(L)$. It satisfies the *no triangles condition* if every finite subset of $V(L)$ that is pairwise joined by edges, is a simplex. More precisely, if $\{v_0, \ldots, v_n\}$ is such that for each $i, j \in \{1, \ldots, n\}$, $\{v_i, v_j\}$ is a 1-simplex of L, then $\{v_0, \ldots, v_n\}$ is a simplex of L.

An alternative name for the condition are the '*no empty simplices*' condition. It is also said that in this case L is *determined by its 1-skeleton*. The point is

Proposition 31 *If simplicial complex, L, is an order complex of some partially ordered set then it is determined by its 1-skeleton.* ■

The proof should be evident.

Geometric group theory contains many other examples of this sort of construction, especially with relation to Coxeter groups. (Perhaps we will return to this later one)

4.3.9 The homotopy type of Vietoris-Volodin complexes

Returning to $V(\mathfrak{H})$, the second complex associated to a pair $(G, \mathcal{H})$, it is possible to extract some homotopy information from it using fairly elementary methods. To go into its structure more deeply we will need to bring more explicitly in the group action of G as well, but that is for later.

The great advantage now is that as we know $N(\mathfrak{H})$ and $V(\mathfrak{H})$ have the same homotopy type (after realisation) so we can use either when working out homotopy invariants. We can also use $N^{simp}(\mathfrak{H})$, or $V^{simp}(\mathfrak{H})$ the corresponding simplicial sets, although, in fact, the Volodin *space* was actually defined as a simplicial set. We will usually leave out the difference between the simplicial complex and the simplicial set as that distinction is largely unnecessary.

If we look at any $gH_i \in \mathcal{H}$, then we have a subcomplex of $V(\mathfrak{H})$ consisting of those $(g_0, \ldots, g_p)$ all of which are in gH_i. In the simplest case, where $g = 1$, this is a copy of $E(H_i)$, and, in general, it is a translated copy of $E(H_i)$, so each forms a contractible subcomplex.

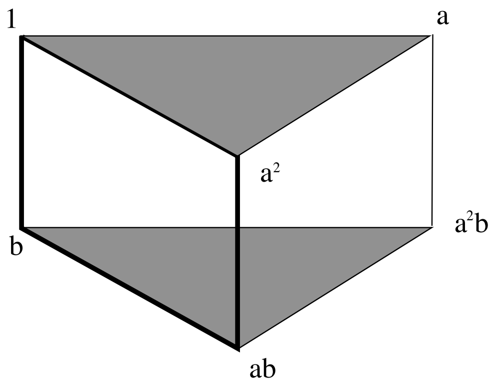
Example: (already considered in section 4.3.1)

$$\begin{aligned} G &= S_3 = (a, b \mid a^3 = b^2 = (ab)^2 = 1), \text{ with } a = (1,2,3), \ b = (1,2); \\ H_1 &= \langle a \rangle = \{1, \ (1,2,3), \ (1,3,2)\}, \\ H_2 &= \langle b \rangle = \{1, \ (1,2)\}; \\ \mathcal{H} &= \{H_1, H_2\} \end{aligned}$$

The intersection diagram given in our earlier look at this example, on page 113, is just the nerve, $N(\mathfrak{H})$, having 5 vertices and 6 edges. The other complex, $V(\mathfrak{H})$, is almost as simple. It has 6 vertices corresponding to the 6 elements of S_3, and each orbit yields a simplex

- $H_1 = \{1, a, a^2\}$ gives a 2-simplex (and three 1-simplices),
- $H_1 b = \{b, ab, a^2 b\}$ also gives a 2-simplex;
- $H_2 = \{1, b\}$ yields a 1-simplex, as do its cosets $H_2 a$ and $H_2 a^2$.

We can clearly see here the contractible subcomplexes mentioned earlier. We have that $V(\mathfrak{H})$ looks like two 2-simplices joined by 1-simplices at the vertices, (see below).



$V(S_3, \{\langle a\rangle, \langle b\rangle\})$

As $N(\mathfrak{H})$ is a connected with 5 vertices and 6 edges, we know $\pi_1 N(\mathfrak{H})$ is free on 2 generators. (The number of generators is the number of edges outside a maximal tree.) This same rank can be read of equally easily from $V(\mathfrak{H})$ as that complex is homotopically equivalent to a bouquet of 2 circles, (i.e., a figure eight). The generators of $\pi_1 V(\mathfrak{H})$ can be identified with words in the free product $H_1 * H_2$ (one such being shown in the picture) and relate to the kernel of the natural homomorphism from $H_1 * H_2$ to S_3. The heavy line in the figure corresponds to a loop at 1 given by

$$1 \xrightarrow{(1,b)} b \xrightarrow{(b,ab)} ab \xrightarrow{(ab,a^2)} a^2 \xrightarrow{(a^2,1)} 1$$

We write $g_0 \xrightarrow{(g_0,g_1)} g_1$ as there is an edge, (g_0, g_1) joining g_0 to g_1 in $V(\mathfrak{H})$. We, thus, have that there is a g and an index i such that $\{g_0, g_1\} \in H_i g$, but the index and the elements are not necessarily uniquely determined. We saw that this means that $g_1 g_0^{-1} \in H_i$, so $g_1 = hg_0$ for some $h \in H_i$, and we could equally well abbreviate the notation to $g_0 \xrightarrow{h} g_1$. Note that the only condition required is that h is in some H_i, so the lack of uniqueness mention above is without importance. In our example, we can redraw the diagram corresponding to the heavier loop and we get

$$1 \xrightarrow{b} b \xrightarrow{a} ab \xrightarrow{b} a^2 \xrightarrow{a} 1$$

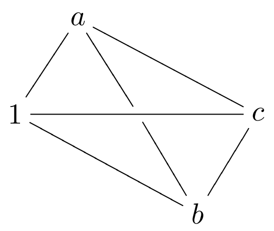
so the loop, representing an element in $\pi_1 N(\mathfrak{H})$, is given by the word $baba \in C_2 * C_3$, which, of course, is in the kernel of the homomorphism from $C_2 * C_3$ to S_3. The reason that this works is clear. Starting at 1, each part of the loop corresponds to a left multiplication either by an element of $H_1 \cong C_3$ or of $H_2 \cong C_2$. We thus get a word in $H_1 * H_2 \cong C_2 * C_3$. As the loop also finishes at 1, we must have that the corresponding word must evaluate to 1 when projected down into S_3.

Note that the two subgroups had simple presentations that combine to give a partial presentation of S_3. The knowledge of the fundamental group, $\pi_1 N(\mathfrak{H})$, then provides information on the 'missing' relations.

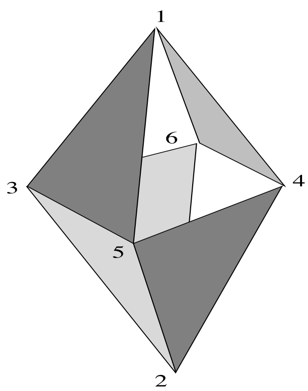
In more complex examples, the interpretation of $\pi_1(V(\mathfrak{H}), 1)$ will be the similar, but sometimes when G has more elements, $N(\mathfrak{H})$ may be easier to analyse than $V(\mathfrak{H})$, but the second may give links with other structure and be more transparent for interpretation. The important idea to retain is that the two complexes give the same information, so either can be used or both together.

Example: $G = K_4$, the Klein 4 group, $\{1, a, b, c\} \cong C_2 \times C_2$, so $a^2 = b^2 = c^2 = 1$ and $ab = c$; $\mathcal{H} = \{H_a, H_b, H_c\}$ where $H_a = \{1, a\}$, etc. Set $\mathfrak{H}_{K4} = (K_4, \mathcal{H})$.

The cosets are $H_a, H_ab, H_b, H_ba, H_c, H_ca$, each with two elements, so $V(\mathfrak{H}_{K4}) \cong$ the 1-skeleton of $\Delta[3]$:



$N(\mathfrak{H}_{K4})$ is "prettier" and a bit more 'interesting': Labelling the cosets from 1 to 6 in the order given above, we have 6 vertices, 12 1-simplices and 4 2-simplices. For instance, $\{1, 3, 5\}$ has the identity in the intersection, $\{1, 4, 6\}$ gives $H_a \cap H_ba \cap H_ca$, so contains a and so on. The picture is of the shell of an octahedron with 4 of the faces removed.



$N(\mathfrak{H}_{K4})$

From either diagram it is clear that $\pi_1\mathfrak{H}_{K4}$ is free of rank 3. Again explicit representations for elements are easy to give. Using $V(\mathfrak{H})$ and the maximal tree given by the edges 1a, 1b and 1c, a typical generating loop would be

$$1 \to a \to b \to 1,$$

i.e., $(1, a, b, 1)$ as the sequence of points. There is an obvious representative word for this, namely

$$1 \xrightarrow{a} a \xrightarrow{c} b \xrightarrow{b} 1\ .$$

In general, any based path at 1 in an $V(G, \mathcal{H})$ will yield a word in $\sqcup\mathcal{H}$, the free product of the family $\mathcal{H}$. We will think of the path as being represented by a (finite) sequence $(f(n))$ of elements in G, linked by transitions, h_i in the various subgroups. Whether or not that representative is unique depends on whether or not there are non-trivial intersections and "nestings" between the subgroups in the family $\mathcal{H}$, since, for instance, if H_i is a subgroup of H_j, then if $f(n) \to f(n+1)$ using $g \in H_i$, it could equally well be taken to be $g \in H_j$. As we have mentioned before, the characteristic of the Vietoris-Volodin spaces, $V(G, \mathcal{H})$, is that there is only one possible *element* of G linking $f(n)$ to the next $f(n+1)$ namely $f(n+1)f(n)^{-1}$, but this may be in several of the H_i. We thus have a strong link between $\pi_1(V(G, \mathcal{H}))$ and $\underset{\cap}{\sqcup}\mathcal{H}$, the 'amalgamated product' of $\mathcal{H}$ over its intersections, and an analysis of homotopy classes will prove (later) that

$$\pi_1(V(G,\mathcal{H}),1) \cong \operatorname{Ker}(\underset{\cap}{\sqcup}\mathcal{H} \to G),$$

since a based path $(g_1, g_2, \cdots, g_n)$ ends at 1 if and only if the product $g_1 \cdots g_n = 1$. These identifications will be investigated more fully shortly.

We note that composites of such 'paths' may involve two adjacent transitions between elements being in the same H_i in which case we can use the rewriting system determined by the contractible $E(H_i)$ to simplify the representatives.

Example: The number of subgroups in $\mathcal{H}$ clearly determines the dimension of $N(\mathfrak{H})$, when $\mathfrak{H} = \mathfrak{H}(G,\mathcal{H})$. Here is another 3 subgroup example.

Take $q8 = \{1, i, j, k, -1, -i, -j, -k\}$ to be the quaternion group, so $i^4 = j^4 = k^4 = 1$, and $ij = k$. Set $H_i = \{1, -1, i, -i\}$ etc., so $H_i \cap H_j = H_i \cap H_k = H_j \cap H_k = \{1, -1\}$ and let $\mathcal{H} = \{H_i, H_j, H_k\}$, and $\mathfrak{H}_{q8} = \mathfrak{H}(q8, \mathcal{H})$.

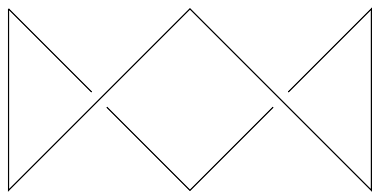
Then $N(\mathfrak{H}_{q8})$ is, as above in Example 4.3.9, a shell of an octahedron with 4 faces missing. Note however that $V(\mathfrak{H}_{q8})$ has 8 vertices and, comparing with $V(\mathfrak{H}_{K4})$, each edge of that diagram has become enlarged to a 3-simplex. It is still feasible to work with $V(\mathfrak{H}_{q8})$ directly, but $N(\mathfrak{H}_{q8})$ gives a clearer indication that

$$\pi_1(\mathfrak{H}_{q8}, 1) \text{ is free of rank 3.}$$

Example: Consider next the symmetric group, S_3, given by the presentation

$$S_3 := (x_1, x_2 \mid x_1^2 = x_2^2 = 1, (x_1x_2)^3 = 1)$$

Take $H_1 = \langle x_1 \rangle$, $H_2 = \langle x_2 \rangle$, so both are of index 3. Each coset intersects two cosets in the other list giving a nerve of form (see below):



so $\pi_1 N(\mathfrak{H}(S_3, \mathcal{H}))$ is infinite cyclic.

Example: The next symmetric group, S_4, has presentation

$$S_4 := (x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = 1, (x_1x_2)^3 = (x_2x_3)^3 = 1, (x_1x_3)^2 = 1).$$

Take $H_1 = \langle x_1, x_2 \rangle$, $H_2 = \langle x_2, x_3 \rangle$, $H_3 = \langle x_1, x_3 \rangle$. H_1 and H_2 are copies of S_3, but H_3 is isomorphic to the Klein 4 group, K_4. Thus there are $4 + 4 + 6$ cosets in all. There are 36 pairwise intersections and each edge is in two 2-simplices. Each vertex is either at the centre of a hexagon or a square, depending on whether it corresponds to a coset of H_1, H_2 or of H_3. There are 24 triangles, and $N(S_4, \mathcal{H})$ is a surface. Calculation of the Euler characteristic gives 2, so this is a triangulation of S^2, the two sphere. (Thanks to Chris Wensley for help with the calculation using GAP.)

The fundamental group of $N(S_4,\mathcal{H})$ is thus trivial and, using the result mentioned above,

$$S_4 \cong \bigsqcup_{\cap} H_i,$$

the coproduct of the subgroups amalgamated over the intersection.

Accepting Proposition 28 for the moment, we can examine an important class of examples.

Example: Some graphs of groups. Let us suppose that $\mathcal{H} = \{H_1, H_2\}$, so just two subgroups of G, then we have

$$H_1 \bigsqcup_{H_1\cap H_2} H_2 \to G.$$

This is an isomorphism if and only if $N(\mathfrak{H})$ is a connnected graph which has trivial fundamental group, thus exactly when $N(\mathfrak{H})$ is a *tree*. The vertices of $N(\mathfrak{H})$ are the cosets in $H_1\backslash G \sqcup H_2\backslash G$ and H_1g_1 and H_2g_2 are connected by an edge if they intersect. This gives us one of the two basic types of a *graph of groups* as defined by Serre, [117, 118],

$$H_1 \underset{H_1\cap H_2}{\text{——————}} H_2$$

corresponding to a free product with amalgamation. Note this does not seem to give us the other basic type of graph of groups which corresponds to an HNN extension. We will see another connection with this theory a bit later or, more exactly, we will see a connection with the generalisation *complexes of groups* due to Corson, [48–50] and Haefliger, [69, 70] and developed extensively in the book by Bridson and Haefliger, [23].

We have now seen, somewhat informally, discussions of the low dimensional homotopy invariants of these two nerves, both in examples and, to some extent, in general. We turn now to more formal calculations of those, and in the process will prove Proposition 28.

We will approach the determination of the invariants in an 'elementary' but reasonably formal way. We will repeat some arguments that we have already seen partially to get everything in the same place, but also to impose some more consistent notation.

The set, $\pi_0(V(G,\mathcal{H}))$, of connected components: The vertex set of $V(G,\mathcal{H})$ is the set of elements of G, so we have to work out when two vertices, g and g', are in the same connected component.

Suppose they are connected by a path, that is a sequence of edges, $(\langle g_0,g_1\rangle, \langle g_1,g_2\rangle, \ldots \langle g_{n-1},g_n\rangle)$, in $V(G,\mathcal{H})$ and for some n. We have that an edge such as $\langle g_0,g_1\rangle$ has $d_0\langle g_0,g_1\rangle = g_1$ and $d_1\langle g_0,g_1\rangle = g_0$ and it is an edge because there is some $H_{\alpha_1}\in\mathcal{H}$ and some $x_1\in G$ such that g_0 and g_1 are in the coset $H_{\alpha_1}x_1$. Of course, this means that there are $h_0,h_1\in H_{\alpha_1}$ with $g_0 = h_0x_1$ and $g_1 = h_1x_1$, hence that $g_0g_1^{-1}\in H_{\alpha_1}$. (Conversely if $g_0g_1^{-1}\in H_{\alpha_1}$, then both g_0 and g_1 are in $H_{\alpha_1}g_1$, so $\langle g_0,g_1\rangle$ is an edge.)

We thus have from our path that there are indices $\alpha_1,\ldots,\alpha_n$ such that $g_{i-1}g_i^{-1}\in H_{\alpha_i}$ for each i, whilst $g = g_0$ and $g' = g_n$. We then note that gg'^{-1} is in $\langle\bigcup\mathcal{H}\rangle$, the subgroup generated by the union of the subgroups in the family $\mathcal{H}$, so, if g and g' are in the same component, then $gg'^{-1}\in\langle\bigcup\mathcal{H}\rangle$.

Conversely, suppose $gg'^{-1} \in \langle \bigcup \mathcal{H} \rangle$, then there is a finite sequence of indices, $\alpha_1, \ldots, \alpha_n$ for some n and elements $h_i \in H_{\alpha_i}$ such that $gg'^{-1} = h_1 h_2 \ldots h_n$. We define $g_0 = g$, $g_i = h_i^{-1} g_{i-1}$ and note that $g_{i-1}, g_i \in H_{\alpha_i} g_i$, thus giving us a path from g to $g_n = h_n^{-1} g_{n-1} = h_n^{-1} \ldots h_1^{-1} g_0 = g'$.

We thus have proved that $\pi_0(V(G, \mathcal{H}))$ is in bijection with $G/\langle \bigcup \mathcal{H} \rangle$, that is the first part of Proposition 28.

The fundamental group, $\pi_1(V(G, \mathcal{H}), 1)$, and groupoid, $\Pi_1(V(G, \mathcal{H}))$: Although $V(G, \mathcal{H})$ comes with a natural choice of basepoint, namely 1, and we will eventually be looking at loops at 1, it is more in tune with our just previous discussion to look at the fundamental groupoid $\Pi_1(V(G, \mathcal{H}))$ rather than the fundamental group $\pi_1(V(G, \mathcal{H}), 1)$ of $V(G, \mathcal{H})$ based at 1. We will sometimes abbreviate $\Pi_1(V(G, \mathcal{H}))$ to $\Pi_1\mathfrak{H}$.

The set of objects of this groupoid will be the vertices of $V(G, \mathcal{H})$ and so are the elements of G, and the set of arrows $\Pi_1\mathfrak{H}(g, g')$ will be the set of homotopy classes of paths from g to g'. We saw that a path from, g to g' corresponds to a finite sequence, $\underline{h} = (h_1, h_2, \ldots, h_n)$, of elements from the various subgroups H_{α_i} in $\mathcal{H}$. It is convenient to write

$$g \xrightarrow{(h_1, h_2, \ldots, h_n)} g' = g \xrightarrow{\underline{h}} g',$$

where $h_n^{-1} \ldots h_1^{-1} g = g'$. We can see that given two composable paths

$$g \xrightarrow{\underline{h}} g' \xrightarrow{\underline{h}'} g'',$$

the defining sequence of the composite is given by the concatenation of the two sequences,

$$\underline{hh'} = (h_1, h_2, \ldots, h_n, h'_1, h'_2, \ldots, h'_m).$$

Remark: This notation is not quite accurate. The $\underline{h}$ does not indicate from where the arrow, so labelled, starts. Of course, it is visually clear, but 'really' we should denote the arrows by $(g, \underline{h})$, so then

$$(g, \underline{h}) \cdot (\underline{h}^{-1} g, \underline{h'}) = (g, \underline{hh'}),$$

or similar. This is clearly a form related to, but not identical, to some sort of 'action groupoid', but that does not quite fit. For a start, it does not give a groupoid as where are the inverses? It does give a category, however. (It is **left for you to check** that $\langle g_0, g_0 \rangle$ is the identity at the 'object' g_0.)

the paths between the vertices are not the actual arrows in the fundamental groupoid $\Pi_1\mathfrak{H}$. For that we need to divide out by relations coming from 2-simplices.

For any simplicial complex or simplicial set, K, one can form the fundamental groupoid, (also called in this context the *edge path groupoid*), by taking the free groupoid on the directed graph given by the 1-skeleton and then dividing out by the 2-simplices. (We will see this several times later; see pages **??**, and **??**. It is the classical edge-path groupoid to be found, for instance, in Spanier's book, [121].) The arrows are sequences of concatenated edges and then, if $\langle v_0, v_1, v_2 \rangle$ is a 2-simplex, we add a 'relation'

$$\langle v_0, v_1 \rangle \langle v_1, v_2 \rangle = \langle v_0, v_2 \rangle,$$

or if you prefer, rewrite rules:

$$\langle v_0, v_1 \rangle \langle v_1, v_2 \rangle \Leftrightarrow \langle v_0, v_2 \rangle.$$

For $\Pi_1\mathfrak{H}$, a 2-simplex in $V(G,\mathcal{H})$ will, of course, be a triple, (g_0, g_1, g_2), of elements of G contained in some $H_\alpha x$. We explore this in detail as before. There will be three elements, h_0, h_1, h_2 in H_α with $g_i = h_i x$ for $i = 0, 1, 2$ and thus $g_i g_j^{-1} \in H_\alpha$, for each i and j.

Dividing out by these relations has several neat consequences which 'control' the paths and their compositions. For instance, working in the simplicial set version of $V(G,\mathcal{H})$, if we have $\langle g_0, g_1\rangle$ in $V(G,\mathcal{H})$, then $\langle g_1, g_0\rangle$ is there as well, and so is $\langle g_0, g_0\rangle$ and as $\langle g_0, g_1, g_0\rangle$ is in $V(G,\mathcal{H})_2$, we have that

$$\langle g_0, g_1\rangle\langle g_1, g_0\rangle = \langle g_0, g_0\rangle,$$

so $\langle g_0, g_1\rangle$ has $\langle g_1, g_0\rangle$ as its inverse. Another important result of these relations is that it allows simplification of the path labelling sequences. Suppose we have a composite path

$$g_0 \xrightarrow{h_1} g_1 \xrightarrow{h_2} g_2$$

which stays more than one step in a given coset, i.e., both h_1 and h_2 are in some H_α. In this case we can clearly replace that path, up to homotopy, that is, modulo the relations, by

$$g_0 \xrightarrow{h_1 h_2} g_2$$

as $\langle g_0, g_1, g_2\rangle$ is a 2-simplex. This means that every arrow in $\Pi_1\mathfrak{H}$ has a representative whose corresponding sequence $\underline{h}$ corresponds to an element of the coproduct (aka free product), $\sqcup H_i$, of the groups in $\mathcal{H}$. This is still not a unique representative however. We may have a situation

$$g_0 \xrightarrow{h_1} g_1 \xrightarrow{h_2} g_2 \xrightarrow{h_3} g_3$$

where $h_1, h_2 \in H_i$ and $h_2, h_3 \in H_j$, so we will have an overlap with $\langle g_0, g_1\rangle\langle g_1, g_2\rangle\langle g_2, g_3\rangle$ rewriting both to $\langle g_0, g_2\rangle\langle g_2, g_3\rangle$ and to $\langle g_0, g_1\rangle\langle g_1, g_3\rangle$, and so we have to *amalgamate* the coproduct over intersections.

Let us be a bit more precise about this. We form up a diagram of the subgroups H_i in $\mathcal{H}$, together with their pairwise intersections, $H_i \cap H_j$. We write $H = \underset{\cap}{\sqcup}\mathcal{H}$ for its colimit.

Definition: Given a family, $\mathcal{H}$, of subgroups of G, its *free product* or *coproduct amalgamated along the intersections* is the colimit, H, specified above.

This group, H, can be given as simple presentation. Take as set of generators a set, $X = \{x_g \mid g \in \bigcup H_j\}$, in bijection with the elements of the union of the underlying sets of subgroups in $\mathcal{H}$, and for relations all $x_{h_1} x_{h_2} = x_{h_1 h_2}$ where h_1 and h_2 are both in some group, H_i, of the family.

The inclusion of each H_j into G gives a cocone on the diagram of groups, so induces a homomorphism, $p : \underset{\cap}{\sqcup}\mathcal{H} \to G$, which will be essential in our description. This homomorphism, p, thus takes a sequence $\underline{h} = (h_1, \ldots, h_n)$ representing some element of H and evaluates it within G mapping it to the product $h_1 \ldots h_n \in G$.

Clearly we have

Proposition 32 *The fundamental groupoid, $\Pi_1\mathfrak{H}$, has for objects the elements of G and an arrow from g to g' is representable, uniquely, by an element h in $\underset{\cap}{\sqcup}\mathcal{H}$ such that $g = p(h)g'$.* ■

The proof is by comparison of the two presentations.

Corollary 6 *There is an isomorphism*

$$\pi_1\mathfrak{H} \cong Ker(p : \underset{\cap}{\sqcup}\mathcal{H} \to G)$$

Proof: The group $\pi_1(V(G,\mathcal{H}),1)$ is the vertex group at 1 of the edge path groupoid, so consists of the $h$in H, which evaluate to 1, since here $g = g' = 1$, i.e. the vertex group is just $Ker\, p$. ■

This means that we have $p : H \to G$, whose 'cokernel', $G/p(H)$, 'is' $\pi_0(V(G,\mathcal{H}))$ and whose kernel is $\pi_1(V(G,\mathcal{H}),1)$.

What about $\pi_2V(\mathfrak{H})$? We will limit ourselves, here, to a special case, and will merely quote a result from the paper of Abels and Holz, [1]. We suppose as always that we are given $(G,\mathcal{H})$ and now assume that we use the standard presentation $\mathcal{P}_j := (X_j : R_j)$ of each H_j. Combining these we get $X = \bigcup X_j$, $R = \bigcup R_j$. We have $\mathcal{H}$ is 2-generating for G if and only if $\mathcal{P} = (X,R)$ is a presentation of G. (That is nice, since it says that there are no hidden extra relations needed, and that corresponds to the intuitions that we were mentioning earlier. There is better to come!) Assuming that $\mathcal{P}$ is a presentation of G, we have a module of identities, $\pi_\mathcal{P}$. We also have all the $\pi_{\mathcal{P}_j}$, the identity modules for each of the presentations, $\mathcal{P}_j$. The inclusions of generators and relations induce morphisms of the crossed modules, $C(\mathcal{P}_j) \to C(\mathcal{P})$, and hence of the modules $\pi_{\mathcal{P}_j} \to \pi_\mathcal{P}$, although here there is the slight complication that this is a morphism of modules over the inclusion of H_j into G, which we will not look further into here. We let $\pi_\mathcal{H}$ be the sub G-module of $\pi_\mathcal{P}$ generated by the images of these $\pi_{\mathcal{P}_j}$. We can think of $\pi_\mathcal{H}$ as the sub-module of $\pi_\mathcal{P}$ consisting of those identities that come from the presentations of the subgroups.

In the above situation, i.e., with standard presentations for the subgroups, we have ([1] Cor. 2.9.)

Proposition 33 *If $\mathcal{H}$ is 2-generating, then there is an isomorphism:*

$$\pi_2(N(\mathfrak{H}) \cong \pi_\mathcal{P}/\pi_\mathcal{H}.$$

■

We should therefore, and in this case at least, interpret $\pi_2(N(\mathfrak{H})$ as telling us about the 2-syzygies that are not due to the presentations of the subgroups. We will give shortly a neat example of this but first would note that this does not interpret the 2-type of $V(\mathfrak{H})$ in general, and that somehow is a lack in the theory as developed so far. Abels and Holz do extend thie away from the standard presentations of the subgroups, but this requires a bit more than we have available at this stage in the notes so will be 'put on hold' until later.

This gives all the easily available data on these Vietoris-Volodin complexes as far as their elementary homotopy information is concerned. We can, and will, extract more later on, but now want to look at the main example for their original introduction.

4.3.10 Back to the Volodin model ...

Our 'more complex family' of section 4.3.3 leads to a link with higher algebraic K-theory in the version developed initially by Volodin. The usual approach, however, uses a slightly different notation and for some of its details ends up looking different, so here we will give the version of that example nearer to that given by, for instance, Suslin and Wodzicki, [122], or Song, [119]. Let, as before, R be an associative ring, and now let σ be a partial order on $\{1, \ldots, n\}$. If i is less that j in the partial order σ, it is convenient to write $i \overset{\sigma}{<} j$. (Note that this means that some of the elements may only be related to themselves and hence are really not playing a role in such a σ.) We will write $PO(n)$ for the set of partial orders of $\{1, \ldots, n\}$.

Definition: We say an $n \times n$ matrix, $A = (a_{ij})$ is *σ-triangular* if, when $i \overset{\sigma}{\not\leq} j$, $a_{ij} = 0$, and all diagonal entries, a_{ii} are 1.

We let $T_n^\sigma(R)$ be the subgroup of $G\ell_n(R)$ formed by the σ-triangular matrices.

Lemma 24 *If $n \geq 3$, $T_n^\sigma(R)$ has a presentation with generators $x_{ij}(a)$, where $i \overset{\sigma}{<} j$ and $a \in R$, and with relations:*

$$x_{ij}(a)x_{ij}(b) = x_{ij}(a+b) \qquad\qquad i \overset{\sigma}{<} j, \quad a, b \in R$$

and

$$[x_{ij}(a), x_{jk}(b)] = x_{ik}(ab) \qquad\qquad i \overset{\sigma}{<} j \overset{\sigma}{<} k, \quad a, b \in R,$$

$$x_{ij}(a)x_{k\ell}(b) = x_{k\ell}(b)x_{ij}(a), \quad i \neq \ell, j \neq k, \;\; a, b \in R.$$

■

Remark: In fact, Kapranov and Saito, [84], mention that, not only is this a presentation of $T_n^\sigma(R)$, but with the addition of the syzygies that they describe (and which up to dimension 2 are those given in our section 4.1.2) gives a complete set of syzygies, of dimension 3.

We can 'stablise' the above, since it σ is a partial order on $\{1, \ldots, n\}$, then it extends uniquely to one on $\{1, \ldots, n+1\}$ by specifying that $n+1$ is related to itself in the extended version, but to no other. (The notation and treatment for this is not itself that 'stable' and some sources do not go into a detailed handling of this point, presumably because it is clear what is going on.) We will write $\mathfrak{T}_n = (G\ell_n(R), \mathcal{T}_n)$, where $\mathcal{T}_n = \{T_n^\sigma(R) \mid \sigma \in PO(n)\}$, and then, letting n 'go to infinity' write $\mathfrak{T}$ for the corresponding system based on $G\ell(R)$ with all σ-triangular subgroups for all partial orders having finite 'support', i.e., in which outside some finite set, (its support), the partial order is trivial.

Proposition 34 *For $n \geq 3$, the subgroup of $G\ell_n(R)$ generated by the union of the $T_n^\sigma(R)$ is $E_n(R)$, the elementary subgroup of $G\ell_n(R)$.*

Proof: This should be more or less clear as, by definition, any elementary matrix is σ-triangular for many σ', and conversely, any $T_n^\sigma(R)$ is given as a subgroup of $E_n(R)$. ■

Corollary 7 *The Volodin nerve, $V(\mathfrak{T})$, has*

$$\pi_0 V(\mathfrak{T}) \cong K_1(R).$$

■

The obvious next question to pose is what $\pi_1(V(\mathfrak{T}),1)$ will be. We know it to be the kernel of $\sqcup_{\cap} T_n^{\sigma}(R) \to E(R)$, and the obvious guess would be that it was Milnor's $K_2(R)$. That's right. Proofs are given in several places in the literature, but usually they require a bit more machinery than we have been assuming up to this point in these notes, so we will not give one of those proofs here. The most usual proofs use the natural action of G on $N(\mathfrak{H})$ and a covering space argument. We will mention this in a bit more detail after we have looked at a sketch proof and will explore aspects of this sort of approach more in a later chapter, but here will attempt to give that sketch proof which, it is hoped, seems more direct and which starts from the descriptions of $\pi_0 V(\mathfrak{T})$ that are consequences of what we have already done above. (We will still need a covering space-type argument, which, since central extensions behave like covering spaces from many points of view, is suggestive of a general approach that is, it seems, nowhere given in the literature with the conceptual simplicity it seems to deserve. Kervaire's treatment of universal central extensions, [85], perhaps goes some way towards what is needed.) We start by looking at paths in $V(\mathfrak{T})$, especially, but not only, those which start at 1. We will be, in part, following Volodin's original treatment in [133] as this is very elementary and 'constructive' in nature. As we said above, he uses covering space intuitions as well, as this seems almost optimal for the identification we need. (Remember that one classical construction of universal covering spaces is from the space of paths that start at the base point, followed by quotienting by fixed end point homotopy as a relation.)

A path in $V(\mathfrak{T})$ as it is of finite length, must live in some $V(\mathfrak{T}_n)$. We thus can represent it by a pair, $(g,\underline{t})$, with $\underline{t}=(t_1,\ldots,t_k)$ for some k, a word with each t_i in some $T_n^{\sigma_i}(R)$, and g in $E_n(R)$ which will be the starting element of the path. (Of course, this representation is not unique, because of the amalgamated subgroups, and we will need to break each t_i up as a product of elementary matrices shortly. The non-uniqueness will be taken account of later on.)

We say that t_i is a *segment* of the path, and that the paths is *elementary* if all the t_is used are elementary matrices.

We now need some 'elementary' linear algebra. We will look at it with respect to the standard maximal linear order on $\{1,\ldots,n\}$ and hence for upper triangular matrices.

Lemma 25 *Let $B=(b_{ij})$ be an upper triangular matrix (with 1s on its diagonal), so b_{ij} is zero if $j<i$. There is a factorisation*

$$B=\prod_{(i,j)} e_{ij}(b_{ij}),$$

with the order of multiplication given by increasing lexicographic order, so $(i,j)>(i_1,j_1)$ if either a) $j>j_1$ of b) $j=j_1$ and $i>i_1$. ■

The proof should be obvious.

We can replace t_k by a path consisting only of elementary matrices (for the ordering σ_i) and with the order of terms given by a lexicographic order in the (i,j)s relative to $\overset{\sigma_i}{<}$. The resulting $t_k=\prod_{(i,j)} e_{ij}(b_{ij})$ and can be 'lifted' to an element

$$\bar{t}_k=\prod_{(i,j)} e_{ij}(x_{ij})\in St_n(R).$$

This element maps down to the element t_k in $E_n(R)$.

Suppose s is a loop, based at 1, in $V(\mathfrak{T})$, but consisting just of elementary matrices in some $T_n^{\sigma_k}(R)$. (We will say s is an elementary loop. We will work with the standard linear order.) As s is a loop at 1, it has a representation as $(1, \underline{s})$, where $\underline{s} = (s_1, \ldots, s_N)$ and the s_ks are in lexicographic order, each s_k is some $e_{ij}(a_ij)$ and, as the path s is a loop, $\prod_{(i,j)} e_{ij}(a_{ij}) = 1$.

Lemma 26 *If s is an elementary loop at 1 in $T_n(R)$, then its lift $\overline{s}$ is $1 \in St_n(R)$.*

Before giving a proof, remember the intuition that seems to be built in Volodin's approach. The $T_n^\sigma(R)$ are seen as patches over which there is a way of lifting paths, so you decompose a long path into bits in the various patches, and then lift them successively. The lifted bits give elements in $St_n(R)$, and 'up there' we have divided out by the homotopy that comes from the relations / rewriting 2-cells. In each patch we expect to get that the lift of s that we are using gives a trivial element (i.e. something like a null-homotopic loop. We thus expect to have to use the presentation of $St(R)$ and, in particular, the embryonic homotopies given by the rewriting 2-cells / relations. As we will see that is exactly what happens.

Proof: We let m be larger than all the i, j involved in the expression for s. (We will generally write $x_{ij}(a)$ etc where a is variable and is really just a 'place marker'.) As $x_{im}(a)x_{kj}(b) = x_{kj}(b)x_{im}(a)$ for $i \neq j$, $k \neq m$, and

$$x_{im}(a)x_{ki}(b) = x_{km}(-ab)x_{ki}(b)x_{im}(a) = x_{ki}(b)x_{km}(-ab)x_{im}(a),$$

we can move all terms of form $x_{im}(a)$ to the right of the product expression for $\overline{s}$. In $St_m(R)$, we thus have

$$\prod_{i<j\leq m} x_{ij}(a) = \prod_{i<j\leq m-1} x_{ij}(a) \cdot \prod_{i<m} x_{im}(a),$$

where, as we said, the a is just a place marker. We thus have that $\overline{s}$ in $St(R)$ can be decomposed as the product of two parts corresponding to loops (down in $E(R)$). These are $\prod_{i<j\leq m-1} x_{ij}(a)$ and $\prod_{i<m} x_{im}(a)$. (As this latter is in the subgroup of $St_m(R)$ generated by the $x_{im}(a)$, this must itself evaluate to 1 as the product does, hence also the other factor must.) Working on the product $\prod_{i<m} x_{im}(a)$ and using the facts firstly that the terms commute with each other by the first rule we recalled above, and then using the first Steinberg relation: $St1 : x_{im}(a)x_{im}(b) = x_{im}(a + b)$, we can now check that this word must itself be trivial as it evaluates to 1.

We now can use backwards induction on m to gradually you get back to the minimal value possible and get the result. ■

Corollary 8 *If s is an elementary loop in some $T_n^\sigma(R)$, then the corresponding lifted word in $St(R)$ is trivial.*

Proof: We have done most of this, except it was in the case of the standard linear order. One can either adapt the above to the general case, or more neatly note that s conjugates, using permutation matrices, to give an element in that linear case. The lifting goes across to $St(R)$ and so the result follows after a bit of checking. ■

Now look at any path in $V(\mathfrak{T})$, starting at 1. Take an elementary representative and examine the initial segment, $1 \xrightarrow{t_1} t_1^{-1}$, so $t_1 \in T_n^{\sigma_1}(R)$. We can lift t_1 to give an element $\bar{t}_1 \in St_n(R)$. This will, in general, depend on the choice of σ_1, but if σ_1' is another possible partial order (i.e.,

$t_1 \in T_n^{\sigma_1}(R) \cap T_n^{\sigma_1'}(R)$, then the resulting two lifts of t_1 will form a 'loop' $\bar{t}_1 \cdot \overline{t'}_1^{-1}$ in $St_n(R)$, but then this loop must be trivial by the lemma and its corollary. We pass to the next 'node' in the path and continue. The next segment does not start at 1, but the argument adapts easily as the corresponding labelling element in the coproduct with amalgamation is all that is used.

This gives that each path s in $V(\mathfrak{T})$ uniquely determines an element $\overline{s}$ in $St(R)$. It is now fairly clear where the argument has to go. The standard classical construction of a universal covering space is via paths starting at some base point 'modulo' fixed endpoint homotopy, so one checks that homotopic paths lift to the same element of $St(R)$. (This is Volodin's Lemma 3.4 of [133], but it is easy to see how it is to go.) Volodin is using the 'patches' given by the $T_n^\sigma(R)$ to lift a path in $E_n(R)$. (This mix of topological intuition with combinatorics and algebra is the starting point of Bak's theory of global actions, [8, 9], that was mentioned earlier.)

It is now feasible to complete the proof à la Volodin, that the universal cover of $V(E_n(R), \{T_n^\sigma(R)\})$ is 'related to' $St_n(R)$, but that is not really satisfactory as it mixes the categories in which we are working. (A simplicial complex is not a group!) We have a more limited aim, namely to note that if we have an element in $\pi_1(V(\mathfrak{T}), 1)$, then we can pick a loop, s, representing it. We can lift s uniquely by lifting over each 'patch' $T_n^\sigma(R)$ that it uses, to obtain an element in $St(R)$, but as it is a loop its evaluation, back down in $G\ell(R)$ will be trivial. (Topologically its endpoint is over the basepoint!) It is in the kernel of the homomorphism from $St(R)$ to $G\ell(R)$, so determines an element of $K_2(R)$. Finally one reverses the argument to say that if $\overline{s} \in K_2(R)$, then it is in the image of this morphism. We have thus given an idea of how Volodin's theorem, below, can be proved, using fairly elementary ideas.

Theorem 7 *(Volodin, [133], Theorem 2)*

$$\pi_1(V(\mathfrak{T}), 1) \cong K_2(R).$$

■

Remark: The usual proofs of this result given in more recent sources tend to use the classifying spaces, $BT_n^\sigma(R)$ together with the induced mappings to $BG\ell(R)$ to obtain

$$\bigcup BT_n^\sigma(R) \to BG\ell(R),$$

which is then shown to give the 'homotopy fibre' of the map to $BG\ell(R)^+$. This does seem slightly too reliant on spatially based methods from homotopy theory and a more purely combinatorial group theoretic or 'rewriting' analysis of the constructions, related to Volodin's original proof, should be possible.

We hope to return to the study of the Volodin model for higher algebraic K-theory later on, but are near to the limit of what can be done with the limited tools at our disposal here, so will put it aside for the moment.

4.3.11 The case of van Kampen's theorem and presentations of pushouts

The above example / case study coming from algebraic K-theory is very rich in its structure and its applications, but *is* complex, so we will return to a simpler situation to indicate the direction that this theory of 'higher generation by subgroups' can lead us to. To motivate this recall the formulation of the classical form of van Kampen's theorem.

Theorem 8 *(van Kampen) Let $X = U \cup V$, where U, V and $U \cap V$ are non-empty, open and arc-wise connected. Let $x_0 \in U \cap V$ be chosen as a base point, then the diagram*

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r, "j_{V*}"] \arrow[d, "j_{U*}"'] & \pi_1(V) \arrow[d, "i_{V*}"] \\ \pi_1(U) \arrow[r, "i_{U*}"'] & \pi_1(X) \end{tikzcd}$$

is a pushout square of groups, where each fundamental group is based at x_0. ∎

Proofs can be found in many places in 'the literature', for instance, in Massey's introduction, [96], or in Crowell and Fox, [52]. A proof of a neat more general form of the result is given in Brown's book, [27]. There the result is given in terms of fundamental groupoids, which is very useful for many applications and several variants are also given there. We may have need for some of these later on, but for the moment what we want is the version in terms of group presentations, cf. [52], page 71, for example. This just translates the above pushout result into one about presentations.

Theorem 9 *(van Kampen: alternative form) Let $X = U \cup V$, etc., be as above. Suppose*

- *that $\pi_1(U, x_0)$ has a presentation, $(\mathbf{X} : \mathbf{R})$,*
- *that $\pi_1(V, x_0)$ has a presentation, $(\mathbf{Y} : \mathbf{S})$,*

and

- *that $\pi_1(U \cap V, x_0)$ has one, $(\mathbf{Z} : \mathbf{T})$,*

then $\pi_1(X, x_0)$ has a presentation,

$$(\mathbf{X} \cup \mathbf{Y} : \mathbf{R} \cup \mathbf{S} \cup \{(\overline{j_{U*}(z)})(\overline{j_{V*}(z)})^{-1} \mid z \in \mathbf{Z}\}),$$

where $\overline{j_{U*}(z)}$ is a word in the free group, $F(\mathbf{X})$ representing $j_{U*}(z)$, and similarly for $\overline{j_{V*}(z)}$. ∎

This form gives a way of calculating a presentation, $\mathcal{P}$, of $\pi_1(X, x_0)$ given presentations of the parts. If we see a presentation as the first part of a recipe to construct a resolution of a group, or alternatively to construct an Eilenberg-Mac Lane space for the group, then this is useful and, of course, is used in courses on elementary algebraic topology to calculate the fundamental groups of surfaces. The obvious points to note are that the we take the union of the two generating sets, $\mathbf{X}$ and $\mathbf{Y}$, to be the generating set of $\pi_1(X, x_0)$, but use the generators in $\mathbf{Z}$ to help form *relations* in the pushout presentation, then we use the union of the two sets of relations to give the other relations (which seems sort of natural). This leaves a query. Whatever happened to the relations in the presentation of $\pi_1(U \cap V, x_0)$? To get some idea of what they do, think along the following somewhat vague lines. As those relations correspond to maps of 2-discs into the complex, $K(\mathcal{P})$, of the presentation, $\mathcal{P}$, used to 'kill' the corresponding words, we have two 2-discs with 'the same' boundary and hence map of a 2-sphere into $K(\mathcal{P})$ with no reason for it being homotopically trivial. This suggests that the relations in $\mathbf{T}$ are going to give homotopical 2-syzygies, and this is the case. It also suggests that to build an Eilenberg-MacLane / classifying space from the presentation, $\mathcal{P}$, we could do worse than take the pushout of the complexes of the various other presentations involved.

It is a good idea to abstract this out a bit away from the van Kampen situation for the moment. We suppose that $G = A *_C B$ is a 'free product with amalgamation', so we can describe G by means of a pushout of groups:

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d, "i'"] \\ A \arrow[r, "j'"'] & G \end{tikzcd}$$

It is a standard result that if i and j are injective, then so are i' and j'.

The van Kampen examples can be too complex to work through, but we can in fact gain some intuition about them from one of the simplest examples of such situations. Consider the trefoil knot group, $G(T_{2,3})$. This has a presentation $(a, b : a^3b^{-2} = 1)$. It is therefore an amalgamated coproduct / pushout of three infinite cyclic groups:

$$\begin{tikzcd} (z : \emptyset) \arrow[r, "j"] \arrow[d, "i"'] & (b : \emptyset) \arrow[d] \\ (a : \emptyset) \arrow[r] & G(T_{2,3}) \end{tikzcd}$$

where $i(z) = a^3$ and $j(z) = b^2$. We note that all the input presentations are with empty sets of relations, yet $G(T_{2,3})$ has a single non-trivial relation. If we took the complexes of each presentation, we would merely have a circle for each, and that of the presentation of $G(T_{2,3})$ has to have a 2-cell in it, hence we can see that the construction of the presentation of $G(T_{2,3})$ does not just result from a 'pushout of presentations'! (In fact, what is needed is a homotopy pushout, or, in more general situations than the pushout of a diagram of group, a homotopy colimit. We will say a bit more on this shortly.) We now return to our general situation.

Our abstracted situation is that we have presentations,$\mathcal{P}_Q = (X_Q : R_Q)$ for $Q = A, B$ and C, and get the corresponding presentation for G, given by the analogue of that in the above discussion. We take complexes $K(\mathcal{P}_Q$ modelling each of the presentations in turn. The morphisms between the groups in the diagram lift give a diagram

$$\begin{tikzcd} K(\mathcal{P}_C) \arrow[r, "j_*"] \arrow[d, "i_*"'] & K(\mathcal{P}_B) \arrow[d, "i'_*"] \\ K(\mathcal{P}_A) \arrow[r, "j'_*"'] & K(\mathcal{P}_G) \end{tikzcd}$$

but as the lifts have to be *chosen*, they are only determined up to homotopy, and this will in general only be a square that is homotopy coherent, i.e., commutative up to a specified homotopy, (see the later discussion in Chapter 5). In fact, as we do not know that i_* and j_* are injective, the result need not be a pushout, so does not tell us much. An alternative is to see what we can construct from the 'corner':

$$\begin{tikzcd} K(\mathcal{P}_C) \arrow[r, "j_*"] \arrow[d, "i_*"'] & K(\mathcal{P}_B) \\ K(\mathcal{P}_A) & \end{tikzcd}$$

from this we can take its 'homotopy pushout' which begins to be more like the square we had. We have not met this construction yet; it is a double mapping cylinder. This would form a cylinder on $K(\mathcal{P}_C)$ and use the maps to glue copies of the other spaces to its two ends. In here, we will be getting a cylinder with the discs corresponding to the relations in $\mathcal{P}_C$ and these will to cylindrical 2-cells in that double mapping cylinder and hence to a potential homotopical 2-syzygy. This will be picked up by the crossed module of that space or better still the crossed complex. An analysis of this can be found in Brown-Higgins-Sivera, [31], starting on page 338. This is based on an earlier paper by Brown, Moore, Porter and Wensley, [33]. (As an exercise, it is **worth looking at the trefoil group from this viewpoint** and to draw what intuitively the mapping cylinder must look like ... as much as this is feasible.)

We have used this discussion above for two main reasons, first to suggest that the situation *naturally* leads to having to take the homotopies seriously and that implies a study of (at least some) homotopy coherence theory, and homotopy colimits in particular. The other reason is that it suggests that it provides a key set of concepts, as yet at a vague intuitive level, to understand more fully the theory of 'higher generation by subgroups' of Abels and Holz, [1]. If we get our group G, and a 1-generating family of subgroups, $\mathcal{H}$, and want to work out the 'syzygies of G', i.e., some combinatorial information to enable a (crossed) resolution or a small model of a $K(G, 1)$ to be formed, then the idea is that by calculating the syzygies of each of the input groups, the n-syzygies of G should involve those of the H_is, but also the $(n-1)$-syzygies of the pairwise intersections, $H_i \cap H_j$, and then, why not, the $(n-2)$-syzygies of the triple intersections, and so on. We certainly do not have the machinery to pursue this here, and so will leave it vague.

(In addition to the above references on the pushout, which use homotopy colimits of crossed complexes over groupoids, the original paper of Abels and Holz, [1], also uses homotopy colimit techniques, but this time with chain complexes. It uses these to prove results on the homological finiteness properties of certain groups. That paper is well worth reading. This use of homotopy colimits is also explored in Stephan Holz's thesis, [75].)

Chapter 5

Homotopy Coherence and Enriched Categories.

We are getting to a point where we need some more powerful insights on homotopy coherence and descent, so in the next few chapters we will examine these topics in some detail. This will give us some useful tools for later use. (These chapters are quite long can be skimmed at first reading, but as the tools will be used later, the material is important for later sections.)

At several points in earlier chapters, we have had to replace colimits by 'pseudo' or 'lax' colimits. We have, especially when 'categorifying', had to replace equality or commutativity in some context, by 'equivalence' or 'coherence'. We have now some experience in handling such ideas and hopefully have built up some intuition, gaining a 'feel' for the general method. It is time now to devote some space to solidifying that intuition a bit further as we will be needing to go in more deeply in future sections.

We will not give a full treatment however as that would take up a lot of space and also would detract from the development of gerbes as such. We will discuss various aspects of the problem and various approaches. Some will involve homotopy theoretic viewpoints, others multiple category theoretic ones. The point is that each approach models certain aspects more transparently than others, so it helps to have a 'multiple model' view. There are possible 'unified models', but they tend to be better handled once the partial approaches - simplicial, homotopy theoretic, n-categorical ones - have been at least met and partially mastered.

5.1 Case study: examples of homotopy coherent diagrams

(Before we get into some examples, it is useful to introduce a bit of terminology that we will use from time to time. If we have a 'diagram' in a category $\mathcal{A}$, then we have, more exactly, some functor, $F : \mathcal{J} \to \mathcal{A}$. We will refer to $\mathcal{J}$ as the 'template' of the diagram, as it gives us the shape of the diagram, that is, what the diagram 'looks like'. We may sometimes give just a graph or more likely a directed graph as a 'template' in which case the corresponding free category on that directed graph will be the domain of the functor. We will also extend the use of 'template' to other similar situations in particular to homotopy coherent diagrams.)

The situation we will start with is a triangular diagram

$$\begin{tikzcd} & K_1 \arrow[dr, "k_{12}"] & \\ K_0 \arrow[ur, "k_{01}"] \arrow[rr, "k_{02}"'] & & K_2 \end{tikzcd}$$

of three spaces or, preferably, simplicial sets, and three maps such that, for the moment, $k_{12} \circ k_{01} = k_{02}$. We can, and will, consider this as a functor

$$K : [2] \to \mathcal{S},$$

where, as always, [2] is the ordinal $\{0 < 1 < 2\}$, considered as a small category. (It is the 'template' for this type of diagram.)

Suppose now that we want to change each K_i to a corresponding object, L_i, which is homotopy equivalent to it. This often occurs when, for instance, the K_is are $K(G, 1)$s, and so have only their fundamental groups non-trivial amongst their homotopy groups. It may be thought useful to replace the K_is by smaller or simpler models that reflect the structure of the $\pi_1(K_i)$s. Suppose, therefore, that we have specified maps

$$\left.\begin{array}{l} f_i : K_i \to L_i \\ g_i : L_i \to K_i \end{array}\right\} \quad i = 0, 1, 2,$$

and homotopies

$$\left.\begin{array}{l} \mathbf{H}_i : Id_{K_i} \simeq g_i f_i \\ \mathbf{K}_i : Id_{L_i} \simeq f_i g_i \end{array}\right\} \quad i = 0, 1, 2.$$

We had a commutative diagram linking the K_is. Can we construct some similar diagram from the L_is? The answer is 'yes, but ...'.

We, of course, need some maps $\ell_{ij} : L_i \to L_j$, and there seems only one possible way of obtaining them in a sensible way, namely, use g to get back to K, go around the K-diagram and then pop back to L using f, i.e., define $\ell_{ij} : L_i \to L_j$ by $\ell_{ij} := (L_i \xrightarrow{g_i} K_i \xrightarrow{k_{ij}} K_j \xrightarrow{f_j} L_j)$. This seems the only way - yet it will not work in general. Yes, these ℓ_{ij}s will exist, but

$$\begin{tikzcd} & L_1 \arrow[dr, "\ell_{12}"] & \\ L_0 \arrow[ur, "\ell_{01}"] \arrow[rr, "\ell_{02}"'] & & L_2 \end{tikzcd}$$

will not commute in general. In fact,

$$\ell_{12} \circ \ell_{01} = f_2 k_{12} g_1 f_1 k_{01} g_0.$$

whilst

$$\ell_{02} = f_2 k_{02} g_0 = f_2 k_{12} k_{01} g_0,$$

so we have $g_1 f_1$ blocking the way! As $Id_{K_1} \simeq g_1 f_1$, $\ell_{02} \simeq \ell_{12} \circ \ell_{01}$, and so the triangle *is* homotopy commutative, but it is more than that since we were told a homotopy $\mathbf{K}_1 : Id_{K_1} \simeq g_1 f_1$, and so have a specific homotopy that does the job, namely $\mathbf{L}_{012} := f_2 k_{12} \mathbf{K}_1 (k_{01} g_0 \times I)$.

$$\begin{tikzcd}
 & L_1 \arrow[dr, "\ell_{12}"] & \\
L_0 \arrow[ur, "\ell_{01}"] \arrow[rr, "\ell_{02}"'] \arrow[rr, phantom, shift left=3ex, "\Uparrow"] & & L_2
\end{tikzcd}$$

Remark: The homotopies we used above went from the identity maps to the composites. We could equally well have written them around the other way. The only difference would be that the arrow in the above diagram would go down instead of up. The conventions here vary from source to source. The above is useful here because it will reflect the cocycle formulae that we have already used, but at other points in our discussion, it will not necessarily be the optimal choice. As homotopies are reversible, it essentially makes no difference here, but it can lead to different formulae and some confusion if this is forgotten.

Now we try to do a slightly harder example. The input this time will be

$$K : [3] \to \mathcal{S},$$

together with $f_i : K_i \to L_i$, $g_i : L_i \to K_i$, $\mathbf{H}_i$, and $\mathbf{K}_i$, for $i = 0, \ldots, 3$. We have maps ℓ_{ij} as before, but also homotopies $\mathbf{L}_{ijk} : \ell_{ik} \simeq \ell_{jk} \circ \ell_{ij}$ for $i < j < k$ within [3], given by $\mathbf{L}_{ijk} := f_k k_{jk} \mathbf{K}_j (k_{ij} g_i \times I)$.

(Any doubts as to why we are going on this excursion into homotopy coherence should be beginning to dissipate by now!) We thus have a tetrahedral diagram

$$\begin{tikzcd}
 & L_1 \arrow[rr, "\ell_{13}"] \arrow[dr, "\ell_{12}"] & & L_3 \\
L_0 \arrow[ur, "\ell_{01}"] \arrow[urrr, "\ell_{03}"] \arrow[rr, "\ell_{02}"'] & & L_2 \arrow[ur, "\ell_{23}"'] &
\end{tikzcd}$$

with homotopies, as above, in each face.

We saw this sort of diagram when we were discussing fibred categories and, in particular, the 3-cocycle condition which mysteriously came out to be written as a square (cf. page **??**). Here also we can analyse our tetrahedral diagram as a square with vertices corresponding to paths through the diagram from L_0 to L_3 and with edges corresponding to the homotopies in the faces. Of course, for instance, $\mathbf{L}_{123} : \ell_{13} \simeq \ell_{23} \circ \ell_{12}$, so it contributes a 'whiskered homotopy' $\mathbf{L}_{123} \circ \ell_{01} : \ell_{13} \circ \ell_{01} \simeq \ell_{23} \circ \ell_{12} \circ \ell_{01}$. (Note we are here being lazy, using the convenient notation $\mathbf{L}_{123} \circ \ell_{01}$ instead of the more exact $\mathbf{L}_{123} \circ (\ell_{01} \times I)$, which, however, is sometimes essential!)

$$\begin{tikzcd}
\ell_{03} \arrow[r, Rightarrow, "\mathbf{L}_{023}"] \arrow[d, Rightarrow, "\mathbf{L}_{013}"'] & \ell_{23}\ell_{02} \arrow[d, Rightarrow, "\ell_{23} \circ \mathbf{L}_{012}"] \\
\ell_{13}\ell_{01} \arrow[r, Rightarrow, "\mathbf{L}_{123} \circ \ell_{01}"'] & \ell_{23}\ell_{12}\ell_{01}
\end{tikzcd}$$

We can compose these homotopies to get two, in general distinct, homotopies from ℓ_{03} to $\ell_{23}\ell_{12}\ell_{01}$, explicitly calculable in terms of $\mathbf{K}_1$ and $\mathbf{K}_2$. (A useful observation here is that the indices 1 and 2 are in the middle of all the homotopies' indices, never 0 or 3, as should be clear from the constructions, so our homotopies use $\mathbf{K}_1$ and $\mathbf{K}_2$, not the others.)

Remark: These can be viewed as defined from $L_0 \times I$ to L_3. This is most easily seen in the topological case as we have an obvious homeomorphism, $[0,1] \cong [0,\frac{1}{2}] \cup [\frac{1}{2},1]$, which allows a neat concatenation of homotopies. It also works well in the simplicial case provided we have the our objects satisfy the Kan condition, i.e., are Kan complexes.

Simplicially the composition of homotopies is done via a choice of filler. We have two maps

$$L_0 \times \Delta[1] \to L_3$$

i.e., two 1-simplices in $\underline{\mathcal{S}}(L_0, L_3)$, which as we saw earlier (cf. page **??**) is the simplicial set of maps of various 'degrees' from L_0 to L_3, given precisely by

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L),$$

in general. From the two composable homotopies, we obtain a map

$$L_0 \times \Lambda^1[2] \to L_3$$

or equivalently a (2,1)-horn

$$\Lambda^1[2] \to \underline{\mathcal{S}}(L_0, L_3).$$

If L_3 is a Kan complex, then so is $\underline{\mathcal{S}}(L_0, L_3)$. (If you have not met the proof, it is worth looking up. You should find it in more or less any text with a section on simplicial homotopy theory.) From our $(2,1)$-horn, we will get a filler:

$$\Delta[2] \to \underline{\mathcal{S}}(L_0, L_3),$$

and the d_i-face of this is a composite homotopy.) Note it is *a*, not *the*, composite homotopy, as we obtained *a* filler by the Kan condition and could not demand it had any special properties such as 'uniqueness'. This point is also valid working with topological homotopies. We conveniently compose homotopies by gluing one copy of a cylinder $X \times I$ to a second one and rescaling. The usual formula looks like

$$H * K(t) = \begin{cases} H(2t) & 0 \le t \le \frac{1}{2} \\ K(2t-1) & \frac{1}{2} \le t \le 1, \end{cases},$$

but this is just one very convenient composite and we could have used many other conventions, for instance, $H(3t)$ for $0 \le t \le \frac{1}{3}$, and $K(3t-2)$ for $\frac{1}{3} \le t \le 1$. Any homeomorphism $h : [0,1] \to [0,2]$ such that $h(0) = 0$ and $h(1) = 2$ will give another composite homotopy.)

That being said, the really neat way to treat this square is ... as a square! We need to specify a 2-fold homotopy, so want a map $\theta : L_0 \times I^2 \to L_3$, which fills the square, i.e., $\theta(x, s, t) \in L_3$ for $(s,t) \in I^2$ and for $x \in L_0$, with

$$\begin{aligned} \theta(x, s, 0) &= \mathbf{L}_{023}(x, s), \\ \theta(x, s, 1) &= \mathbf{L}_{123}(\ell_{01}(x), s), \\ \theta(x, 0, t) &= \mathbf{L}_{013}(x, t), \\ \theta(x, 1, t) &= \ell_{23}\mathbf{L}_{012}(x, t). \end{aligned}$$

In the topological case, such a θ would need, of course, to be continuous, but would then be a suitable level 2 homotopy, $\mathbf{L}_{0123}$, completing our solution. We have not said how to construct this θ, but you have all the necessary machinery to do so. It only uses the elements of the data that have already been given. Its construction is **quite useful to do yourselves**, as it shows you how the low dimensional homotopies combine quite simply to give the level 2 homotopy that is needed. It uses a bit of topology, but only in a minimal way.

If we need a simplicial analogue of this, then we would need $\mathbf{L}_{0123} \in \mathcal{S}(L_0 \times \Delta[1]^2, L_3)$. Our simplicial mapping space, $\underline{\mathcal{S}}(L_0, L_3)$, initially looks slightly wrong for this since we need two 2-simplices with one matching common d_1-face to get $\Delta[1]^2$ and all the simplices in $\underline{\mathcal{S}}(L_0, L_3)$ have form $L_0 \times \Delta[n] \to L_3$. In fact this is easy to get around. The category of simplicial sets is Cartesian closed with its internal mapping object given exactly by this $\underline{\mathcal{S}}(K, L)$ construction, so we have, for each triple, K, L, M, of simplicial sets;

$$\underline{\mathcal{S}}(K \times L, M) \cong \underline{\mathcal{S}}(K, \underline{\mathcal{S}}(L, M)).$$

(If you are not familiar with Cartesian closed categories, then do glance at a suitable survey article or category theory textbook, e.g. [21]. The Wikipedia article on the subject will also give you some basic facts and ideas about the concept. You should also consult the n-Lab.)

We can use this isomorphism to convert our desired level 2 homotopy into a simplicial map

$$\Delta[1]^2 \to \underline{\mathcal{S}}(L_0, L_3).$$

(For formalities sake, it may be better to think of $\mathbf{L}_{0123}$ as being

$$\Delta[1]^2 \times L_0 \to L_3$$

instead of as having domain $L_0 \times \Delta[1]^2$.)

This is using the simplicially enriched category structure of $\mathcal{S}$, and allows us to produce and interpret a similar construction in many other simplicially enriched contexts. To do this we will need some more elements of the notions of simplicially enriched categories, also called $\mathcal{S}$-categories. These are just one of the ways of encoding *homotopy coherence*, but they fit neatly into our general approach. Other related concepts would include dg-categories that is, differential graded categories, which are categories enriched over the category of chain complexes. We will have a look at these later.

5.2 Simplicially enriched categories

These are, intuitively, just categories with simplicial 'hom-sets'. We will also call them *$\mathcal{S}$-categories.*

5.2.1 Categories with simplicial 'hom-sets'

We assume we have a category, $\mathcal{A}$, whose objects will often be denoted by lower case letter, x, y, z, ..., at least in the generic case, and for each pair of such objects, (x, y), a simplicial set, $\mathcal{A}(x, y)$, is given. For each triple x, y, z of objects of $\mathcal{A}$, we have a simplicial map, called *composition*,

$$\mathcal{A}(x, y) \times \mathcal{A}(y, z) \longrightarrow \mathcal{A}(x, z);$$

and for each object x, a map,

$$\Delta[0] \to \mathcal{A}(x, x),$$

that 'names' or 'picks out' the 'identity arrow' in the set of 0-simplices of $\mathcal{A}(x,x)$. This data is to satisfy the obvious axioms, associativity and identity, suitably adapted to this situation.

Definition: Such a set-up, as detailed above, will be called a *simplicially enriched category* or, more simply, an $\mathcal{S}$*-category.*

Enriched category theory is a well established branch of category theory. It has many useful tools and not all of them have yet been explored for the particular case of $\mathcal{S}$-categories and its applications in homotopy theory.

Warning: As we have mentioned before, some authors use the term 'simplicial category' for what we have termed a simplicially enriched category. There is a close link with the notion of simplicial category that is consistent with usage in simplicial theory *per se*, since any (small) simplicially enriched category can be thought of as a simplicial object in the 'category of categories', but a simplicially enriched category is not just a simplicial object in the 'category of categories' and not all such simplicial objects correspond to such enriched categories. That being said, that usage need not cause problems provided you are aware of the usage in the paper to which reference is being made.

5.2.2 Examples of $\mathcal{S}$-categories

We have seen the first example several times before, but will repeat it for convenience:

(i) $\mathcal{S}$, the category of simplicial sets:
here
$$\underline{\mathcal{S}}(K,L)_n := \mathcal{S}(\Delta[n]\times K, L);$$
Composition : for $f\in \underline{\mathcal{S}}(K,L)_n$, $g\in\underline{\mathcal{S}}(L,M)_n$, so $f:\Delta[n]\times K\to L$, $g:\Delta[n]\times L\to M$,
$$g\circ f := (\Delta[n]\times K \stackrel{diag\times K}{\longrightarrow} \Delta[n]\times\Delta[n]\times K \stackrel{\Delta[n]\times f}{\longrightarrow} \Delta[n]\times L \stackrel{g}{\to} M);$$
Identity : $id_K : \Delta[0]\times K \stackrel{\cong}{\to} K$.

Notational remark: Perhaps a word on notation is needed here. Above we have used $\mathcal{S}(\Delta[n]\times K, L)$, but as the product is symmetric, we could equally well have used $\mathcal{S}(K\times \Delta[n], L)$, and although in writing these notes I have tried to be consistent for the first of these, there will certainly be instances of the second convention that have crept in as both are used in the source material that I have used! It makes no real difference to the theory, but can make a difference to the formulae. Similar notational conventions, and similar probable errors in the notation, apply to the other examples below.

(ii) Top, 'the' category of spaces (of course, there are numerous variants but you can almost pick whichever one you like as long as the constructions work):
$$\underline{Top}(X,Y)_n := Top(\Delta^n\times X, Y).$$
Composition and identities are defined more or less as in (i).

If our favourite category, Top, of topological spaces has mapping spaces, Y^X, so is itself Cartesian closed, then $\underline{Top}(X,Y)$ can be identified with $Sing(Y^X)$, and this is also true if Y^X exists in Top for some pair of spaces X and Y, even if not all such pairs may have this property.

(iii) For each X, $Y \in Cat$, the category of small categories, then we similarly get $\underline{Cat}(X,Y)$,

$$\underline{Cat}(X,Y)_n = Cat([n] \times X, Y).$$

We leave the other structure up to the reader.

Of course, Cat *is* Cartesian closed and $\underline{Cat}(X,Y) = Ner(Y^X)$, up to isomorphism.

(iv) Crs, the category of crossed complexes: see section 3.1, for the definitions and additional references, [81] for some introductory background, and Tonks, [127] for a more detailed treatment of the simplicially enriched category structure;

$$\mathsf{Crs}(A,B) := Crs(\pi(n) \otimes C, D).$$

Composition has to be defined using an approximation to the identity, again see [127]. (As mentioned before, the forthcoming book by Brown, Higgins and Sivera, [31] contains a coherent exposition of most of the theory of crossed complexes.)

(v) Ch^+_K or, more expansively, $Ch^+(K\text{–}Mod)$, the category of positive chain complexes of modules over a (commutative) ring K. Details are left to **the reader**, or follow from the Dold-Kan theorem and example (vi) below. We will examine this in more detail later on, but will also look at a different enrichment for this category.

(vi) $Simp.K\text{-}Mod$, the category of simplicial K-modules. The structure uses tensor product with the free simplicial K-module on $\Delta[n]$ to define the 'hom' and the composition, so is very much like (i). It is better viewed as being enriched over itself and we will examine it from that viewpoint slightly later.

(vii) Any simplicial monoid is a simplicially enriched category, so also any simplicial group is one. Of course, they only have a single object. Conversely an $\mathcal{S}$-category that has a single object only is a simplicial monoid. The multiplication in the simplicial monoid is the composition in the category etc.

(viii) Any category, C, will give us a $\mathcal{S}$-category, namely the corresponding *trivially enriched* or *locally discrete* $\mathcal{S}$-category. This leads to:

Definition: A $\mathcal{S}$-category, $\mathcal{B}$, is *locally discrete* or, equivalently, *trivially enriched* if each $\mathcal{B}(x,y)$ is a discrete simplicial set.

(ix) Any 2-category, C, will give us an $\mathcal{S}$-category. In fact, a 2-category is precisely a Cat-enriched category, so each 'hom' is a small category. In more detail, suppose C is a 2-category and x, y and z are objects, then the composition

$$c_{x,y,z} : C(x,y) \times C(y,z) \to C(x,z)$$

is a functor. The obvious way to get a simplicial set from $C(x,y)$ is to apply the nerve functor. We let $C^\Delta(x,y) = Ner(C(x,y))$ and we use the fact that we have already noted, that the nerve functor preserves products, then we define the $\mathcal{S}$-category, C^Δ, by the above simplicial 'homs' with composition

$$C^\Delta(x,y) \times C^\Delta(y,z) \cong Ner(C(x,y) \times C(y,z)) \stackrel{Ner(c_{x,y,z})}{\longrightarrow} Ner(C(x,z)) \cong C^\Delta(x,z).$$

The identities look after themselves; associativity and unit axioms are then easily checked. In fact, as the nerve functor *embeds Cat* as a subcategory of $\mathcal{S}$, the resulting $\mathcal{S}$-category is really just the original 2-category in disguise.

(x) We saw in section **??** how to construct a simplicially enriched groupoid, GK, from a simplicial set, K. The terminology *is* consistent. Recall that the set of objects of GK was the set of vertices of K itself and that there were two maps, source and target, given by iterated face maps to K_0, (cf. page **??**). To rewrite GK as a simplicially enriched category, we just take, for objects, x and y of GK, $GK(x,y)_n$ to be the set of arrows in GK_n that start at x and have target y. The composition in GK_n works by construction and all this is compatible with face and degeneracy maps. (The details **should be looked at a bit** as it is very often useful to be able to swap between the two ways of viewing GK. Thinking of the Dwyer-Kan loop groupoid as a simplicially enriched category is akin to thinking of a group G as a small category, so this is central to the 'categorification' story.)

(xi) An important set of examples of nice small $\mathcal{S}$-categories is given by the simplicially enriched category versions of the simplices. These are built from the ordered sets $[n] = \{0 < 1 < \ldots < n\}$ and will be denoted $S[n]$. We will give two equivalent definitions of them, one simple one here, another shortly using a comonadic resolution. The latter is very useful for linking the construction with the cohomology of categories, but the first is very pretty and simple and is easier to understand.

First note that if i and j are in $[n]$, then there are no paths from i to j if $i > j$, but if $i \leq j$, there are 2^{j-i} such paths. (Experiment a bit with simple examples if you do not see this.) More precisely, a path in a category $\mathcal{C}$ from an object, x, to an object, y, is a sequence of arrows in $\mathcal{C}$ joining the two objects:

$$x = c_0 \stackrel{a_1}{\to} c_1 \stackrel{a_2}{\to} \ldots \stackrel{a_k}{\to} c_k = y.$$

It thus determines a functor $a : [k] \to \mathcal{C}$ and, at this stage incidently, a simplex of $Ner(\mathcal{C})$. As $[n]$ is a totally ordered set, each (non-degenerate) such path from i to j is specified just by the set of intermediate objects, (as there are unique arrows between them so there is no choice of the a_ms). It is now clear that there are $j - i - 1$ intermediate elements, between i and j, and so 2^{j-i-1} such paths including the direct path that corresponds to the empty set of intermediate objects. The combinatorial structure of the partially ordered set of such paths is clearly that of $\{0 < 1\}^{j-i-1}$, as each path corresponds to a subset of the intermediate objects of $[n]$. The nerve of this partially ordered set is $\Delta[1]^{j-i}$. If $i \leq j \leq k$, we can define a composition pairing

$$\Delta[1]^{j-i-1} \times \Delta[1]^{k-j-1} \to \Delta[1]^{k-i-1}$$

given by sending a pair consisting of a subset A of $\{i, \ldots, j\}$ and a subset B of $\{j, \ldots, k\}$ to $A \cup \{j\} \cup B$. Note the inclusion of $\{j\}$. It will always be there in that composite. (Here we are working in several contexts at once, paths, subsets of sets of intermediate elements, and simplicial mappings, so it may pay to pause and check details such as compatibility with face and degeneracy maps etc., just to make sure your intuition on what is happening here, and why it works, is up to speed.)

Definition: Let $S[n]$ be the $\mathcal{S}$-category having the same objects as the category $[n]$, with $S[n](i,j)$ empty if $j < i$ and isomorphic to $\Delta[1]^{j-i-1}$ if not, and with the above composition pairing as the composition. We will call $S[n]$ the *Scategorical@$\mathcal{S}$-categorical n-simplex*.

(xii) In general any category of simplicial objects in a 'nice enough' category has a simplicial enrichment, although the general argument that gives the construction does not always make the structure as transparent as it might be.

Proposition 35 *If $\mathcal{A}$ is any category, $Simp(\mathcal{A}) = \mathcal{A}^{\Delta^{op}}$ is an $\mathcal{S}$-category.*

Proof: Let K to be any simplicial set, then Δ/K is the comma category with objects $([n],x)$ with $x \in K_n$ and morphisms $\mu : ([n],x) \to ([m],y)$ being those $\mu : [n] \to [m]$ in Δ such that $K(\mu)(y) = x$. There is a forgetful functor

$$\delta_K : \Delta/K \to \Delta, \qquad\qquad \delta_K([n],x) = x.$$

Now given $X, Y \in Simp(\mathcal{A})$, define

$$Simp(\mathcal{A})(X,Y)_n = NatTrans(X\delta^{op}_{\Delta[n]}, Y\delta^{op}_{\Delta[n]})$$

■

Several times above we have use a notational convention that can be very useful. If a category, $\mathcal{A}$, is to be regarded both as an ordinary category and a simplicially enriched one, there arises a problem of what notation to use for the two types of hom-object. One simple and quite effective solution is to use $\mathcal{A}(A,B)$ if thinking of the *set* of morphisms and an underlined version $\underline{\mathcal{A}}(A,B)$ if it is the simplicial set of morphisms that we mean. Then it is also natural to refer to the basic category as $\mathcal{A}$ and the $\mathcal{S}$-enriched version as $\underline{\mathcal{A}}$. We probably have not been consistent about this, but will try!

There is an evident notion of $\mathcal{S}$-enriched functor, so we get a category of 'small' $\mathcal{S}$-categories, denoted $\mathcal{S}$–Cat. Of course, some of the above examples are not 'small'. (With regard to 'smallness', although sometimes a smallness condition is essential, one can often ignore questions of smallness and, for instance, consider simplicial 'sets' where actually the collections of simplices are not truly 'sets' (depending on your choice of methods for handling such foundational questions).)

5.2.3 From simplicial resolutions to $\mathcal{S}$-categories

The construction of $S[n]$ from $[n]$ is an example of a general construction for any small category. One can approach it via paths as we did above or via a free category construction. This latter approach has the advantage that it emphasises the link between the constructions of the categorical approach to homotopy coherence and the constructions of categorical cohomology theory, as exemplified by the comonadic resolution construction that we used earlier in a particular case, cf. section 3.5.3, page 78. It is therefore useful to present both approaches.

The forgetful functor, $U : Cat \to DGrph_0$, has a left adjoint, F. Here $DGrph_0$ denotes the category of directed graphs with 'identity loops', so U forgets just the composition within each small category, but remembers that certain loops are special 'identity loops'. The free category functor here takes, between any two objects, all strings of composable *non-identity* arrows that start at the first object and end at the second. One can think of F identifying the old identity arrow at an object x with the empty string at x.

This adjoint pair gives a comonad on Cat in the usual way, and hence a functorial simplicial resolution, as we saw on page 78. Here we will use the alternative form of the construction. This takes the face and degeneracy maps in the opposite direction, but is otherwise more or less completely equivalent. We will denote this, for a small category $\mathbb{A}$, by $S(\mathbb{A}) \to \mathbb{A}$. In more detail, we write $L = FU$ for the functor part of the comonad, the unit of the adjunction, $\eta : Id_{DGrph_0} \to UF$, gives the comultiplication, $F\eta U : L \to L^2$, and the counit of the adjunction gives $\varepsilon : FU \to Id_{Cat}$, that is, $\varepsilon : L \to Id$. Now, for $\mathbb{A}$ a small category, set $S(\mathbb{A})_n = L^{n+1}(\mathbb{A})$ with face maps $d_i : L^{n+1}(\mathbb{A}) \to L^n(\mathbb{A})$ given by $d_i = L^i\varepsilon L^{n-i}$, and similarly for the degeneracies, which use the comultiplication in an analogous formula. (Note that there are two conventions possible here. The other will use $d_i = L^{n-i}\varepsilon L^i$. The only effect of such a change is to reverse the direction of certain 'arrows' in diagrams. The two simplicial structures are 'dual' to each other. The difference is exactly that which we noted when we first wrote the homotopy coherent triangle in our first example.)

This $S(\mathbb{A})$ is a simplicial object in Cat, $S(\mathbb{A}) : \mathbf{\Delta}^{op} \to Cat$, so does not immediately gives us a simplicially enriched category, however its simplicial set of objects is constant because U and F took note of the identity loops.

In more detail, let $ob : Cat \to Sets$ be the functor that picks out the set of objects of a small category, then $ob(S(\mathbb{A})) : \mathbf{\Delta}^{op} \to Sets$ is a constant functor with value the set $ob(\mathbb{A})$ of objects of $\mathbb{A}$. More exactly, it is a discrete simplicial set, since all its face and degeneracy maps are bijections. Using those bijections to identify the possible different sets of objects, yields a constant simplicial set where all the face and degeneracy maps are identity maps, i.e., we do now have a *constant* simplicial set of objects.

Lemma 27 *Let $\mathcal{B} : \mathbf{\Delta}^{op} \to Cat$ be a simplicial object in Cat such that $ob(\mathcal{B})$ is a constant simplicial set with value B_0, say. For each pair $(x, y) \in B_0$, let*

$$\mathcal{B}(x, y)_n = \{\sigma \in \mathcal{B}_n \mid dom(\sigma) = x, codom(\sigma) = y\},$$

where, of course, dom *refers to the domain function in $\mathcal{B}_n$, similarly for* codom.

(i) The collection $\{\mathcal{B}(x, y)_n | \ n \in \mathbb{N}\}$ has the structure of a simplicial set, $\mathcal{B}(x, y)$, with face and degeneracies induced from those of $\mathcal{B}$.

(ii) The composition in each level of $\mathcal{B}$ induces

$$\mathcal{B}(x, y) \times \mathcal{B}(y, z) \to \mathcal{B}(x, z).$$

Similarly the identity map in $\mathcal{B}(x, x)$ is defined as id_x, the identity at x in the category $\mathcal{B}_0$.

(iii) The resulting structure is an $\mathcal{S}$-enriched category, that will also be denoted $\mathcal{B}$. ■

The proof is just a matter of checking formulae, and is **left to the reader**.

In particular, this shows that $S(\mathbb{A})$ is a simplicially enriched category. The augmentation of the comonadic resolution yields an $\mathcal{S}$-functor, denoted $d_0 = \eta := \eta_{\mathbb{A}} : S(\mathbb{A}) \to \mathbb{A}$, from $S(\mathbb{A})$ to the locally discrete $\mathcal{S}$-category corresponding to $\mathbb{A}$. (The d_0 notation is useful if considering the whole structure as enriched over *augmented simplicial sets*, .)

Definition: For a small category $\mathbb{A}$, the $\mathcal{S}$-category $S(\mathbb{A})$ is the *free $\mathcal{S}$-category resolving* $\mathbb{A}$ The $\mathcal{S}$-functor $\eta := \eta_{\mathbb{A}} : S(\mathbb{A}) \to \mathbb{A}$ is the *augmentation of this resolution.*

The description of the simplices in each dimension of $S(\mathbb{A})$ that start at a and end at b is intuitively quite simple. The arrows in the category, $L(\mathbb{A})$, correspond to strings of symbols representing non-identity arrows in $\mathbb{A}$ itself, those strings being 'composable' in as much as the domain of the i^{th} arrow must be the codomain of the $(i-1)^{th}$ one, and so on. Because of this we have:

- $S(\mathbb{A})_0$ consists exactly of such composable chains of maps in $\mathbb{A}$, none of which is the identity;
- $S(\mathbb{A})_1$ consists of such composable chains of maps in $\mathbb{A}$, none of which is the identity, together with a choice of bracketting;
- $S(\mathbb{A})_2$ consists of such composable chains of maps in $\mathbb{A}$, none of which is the identity, together with a choice of two levels of bracketting;
- ... and so on.

Face and degeneracy maps remove or insert brackets, but care must be taken when removing innermost brackets as the compositions that can then take place can result in chains with identities, which then need removing, see [42], that is why the comonadic description is so much simpler, as it manages all that itself.

To understand $S(\mathbb{A})$ in general, it pays to examine the simplest few cases. The key cases are when $\mathbb{A} = [n]$, the ordinal $\{0 < \ldots < n\}$ considered as a category as before. We gave these earlier from the other viewpoint, so how do they look from the comonadic one? This sheds light on the links between the two approaches.

The cases $n = 0$ and $n = 1$ give no surprises:

- $S[0]$ has one object 0 and $S[0](0,0)$ is isomorphic to $\Delta[0]$, as the only simplices are degenerate copies of the identity.
- $S[1]$ likewise has a trivial simplicial structure, being just the category $[1]$ considered as an $\mathcal{S}$-category.
- Things do get more interesting at $n = 2$. The key here is the identification of $S[2](0,2)$. There are two non-degenerate strings or paths that lead from 0 to 2, so $S[2](0,2)$ will have two vertices. The bracketted string $((01)(12))$ on removing inner brackets gives (02) and outer brackets, $(01)(12)$, so represents a 1-simplex,

$$(02) \xrightarrow{((01)(12))} (01)(12),$$

Other simplicial homs are all $\Delta[0]$ or empty. It thus is possible to visualise $S[2]$ as a copy of $[2]$ with a 2-cell going towards the top:

$$\begin{tikzcd} & 1 \arrow[dr, "(12)"] & \\ 0 \arrow[ur, "(01)"] \arrow[rr, "(02)"'] & \arrow[u, phantom, "\Uparrow"] & 2 \end{tikzcd}$$

- The next case $n = 3$ is even more interesting: $S[3](i,j)$ will be (i) empty if $j < i$, (ii) isomorphic to $\Delta[0]$ if $i = j$ or $i = j-1$, (iii) isomorphic to $\Delta[1]$, by the same reasoning as we just used, for $j = i+2$ and that leaves $S[3](0,3)$. This is a square, $\Delta[1]^2$, as follows:

$$\begin{tikzcd} (03) \arrow[r, "((02)(23))"] \arrow[d, "((01)(13))"'] \arrow[dr, "diag"] \arrow[d, phantom, shift left=4ex, "a"] & (02)(23) \arrow[d, "((01)(12))((23))"] \arrow[d, phantom, shift right=4ex, "b"] \\ (01)(13) \arrow[r, "((01))((12)(23))"'] & (01)(12)(23) \end{tikzcd}$$

where the diagonal $diag = ((01)(12)(23))$, $a = (((01))((12)(23)))$ and $b = (((01)(12))((23)))$. (It is instructive to check that this is correct, firstly because I may have slipped up (!) as well as seeing the mechanism in action. Removing the outermost brackets is d_0, and so on.)

- The case of $S[4]$ is worth doing. (Yes, that means it is suggested as an **exercise**. As might be expected, $S[4](0,4)$ is a cube.)

The simplicial resolution construction of $S(\mathbb{A})$ from $\mathbb{A}$ was cross referenced to our earlier use of comonadic simplicial resolutions for groups and the link of that with cohomology, see page 78. So as to investigate the link between the two instances of this that we have seen, it is useful to look at a special case of the $\mathcal{S}$-construction, namely when the given small category is a monoid and, in particular, when it is a group.

Let $\mathbb{A}$ be a monoid, thought of as a small category with a single object. The adjoint pair of functors,

$$U : Cat \rightleftarrows DGrph_0 : F\ ,$$

restricts to the category of monoids on the one hand and to that, $Sets_0$, of pointed sets on the other:

$$U : Mon \rightleftarrows Sets_0 : F\ .$$

The basic step in the construction is that of forming the free monoid on the set of the non-identity elements of a monoid, and so the bracketing terminology works well still in this particular situation.

We thus have that $S(\mathbb{A})$ is a simplicial monoid in the ordinary sense of the term. If $\mathbb{A}$ is actually a group rather than 'merely' a monoid, then $S(\mathbb{A})$ is still only a simplicial monoid, but for any $g \in \mathbb{A}$, there are 'generators' $\langle g \rangle$ and $\langle g^{-1} \rangle$ in $S(\mathbb{A})_0$ and a 1-simplex, $(\langle g \rangle, \langle g^{-1} \rangle)$ in $S(\mathbb{A})_1$. We can calculate the vertices on the two ends of this: as $d_0 = \varepsilon T$ and $d_1 = T\varepsilon$,

$$d_0(\langle g \rangle, \langle g^{-1} \rangle) = \langle g \rangle \langle g^{-1} \rangle,$$

and

$$d_0(\langle g\rangle, \langle g^{-1}\rangle) = 1,$$

since $\varepsilon(\langle g\rangle, \langle g^{-1}\rangle) = 1_{\mathbb{A}})$. The 1-simplex thus looks like

$$1 \to \langle g\rangle\langle g^{-1}\rangle.$$

Of course, there is another one from 1 to $\langle g\rangle\langle g^{-1}\rangle$. As $S(\mathbb{A})_0$ is a free monoid, we do not have elements such as $\langle g\rangle^{-1}$ around and so do not get a corresponding 1-simplex *ending* at 1.

Remark: The history of this S-construction is interesting. A variant of it, but with topologically enriched categories as the end result, is in the work of Boardman and Vogt, [20], and also in Vogt's paper, [132]. Segal's student Leitch used a similar construction to describe a homotopy commutative cube (actually a *homotopy coherent cube*), cf. [88], and this was used by Segal in his famous paper, [116], under the name of the 'explosion' of $\mathbb{A}$. All this was still in the topological framework and the link with the comonad resolution was still not in evidence.

Although it seems likely that Kan knew of this link between homotopy coherence and the comonadic resolutions by at least 1980, (cf. [58]), the construction does not seem to appear in his work with Dwyer as being linked with coherence until much later. Cordier made the link explicit in [42] and showed how Leitch and Segal's work fitted in to the pattern. His motivation was for the description of homotopy coherent diagrams of topological spaces. Other variants were also apparent in the early work of May on operads, and linked in with Stasheff's work on higher associativity and commutativity 'up to homotopy', and it would be possible to write a whole course on those and not to stray too far from our theme of non-abelian cohomology either.

Cordier and Porter, [43], used an analysis of a locally Kan simplicially enriched category involving this construction to prove a generalisation of Vogt's theorem on categories of homotopy coherent diagrams of a given type. (We will return to this aspect a bit later in these notes, but an elementary introduction to this theory can be found in [81].) Finally Bill Dwyer, Dan Kan and Jeffrey Smith, [60], introduced a similar construction for an $\mathbb{A}$ which is an $\mathcal{S}$-category to start with, and motivated it by saying that $\mathcal{S}$-functors with domain this $\mathcal{S}$-category corresponded to ∞-*homotopy commutative* $\mathbb{A}$-*diagrams*, yet they do not seem to be aware of the history of the construction, and do not really justify the claim that it does what they say. Their viewpoint is however very important as, basically, within the setting of Quillen model category structures, this provides a cofibrant replacement construction. We will look at cofibrant replacements in another context later on in this chapter. (If you want to check up on this idea now, a good source is Hovey's book, [76].) Of course, any other cofibrant replacement could be substituted for it and so would still allow for a description of homotopy coherent diagrams in that context. This important viewpoint can also be traced to Grothendieck's *Pursuing Stacks*, [67].

The extension of the construction in [60], although simple to do, is often useful and so will be outlined next.

If $\mathbb{A}$ is already a $\mathcal{S}$-category, think of it as a simplicial category, then applying the S-construction to each $\mathbb{A}_n$ will give a bisimplicial category, i.e., a functor $S(\mathbb{A}) : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to Cat$. Of this we take the diagonal, so the collection of n-simplices is $S(\mathbb{A})_{n,n}$, and, by noticing that the result has a constant simplicial set of objects, then apply the lemma.

Before leaving this construction, let us just comment that if we had used the other formulae for the simplicial resolution, the only difference would be that the higher dimensional arrows would be reversed in direction, so that, for instance, in $S[2]$, we would have had the arrow going from the composite of the d_2 and the d_0 *to* the d_1-face, not the other way around.

5.3 Structure

As one can 'do' homotopy theory with simplicial sets, one can adapt that theory to give a basic homotopy theory in any $\mathcal{S}$-category. Of course, some of these homotopy theories will be richer than others.

5.3.1 The 'homotopy' category

If $\mathcal{C}$ is an $\mathcal{S}$-category, we can form a category $\pi_0\mathcal{C}$ with the same objects and having

$$(\pi_0\mathcal{C})(X,Y) = \pi_0(\mathcal{C}(X,Y)).$$

This is known as the *homotopy category of the $\mathcal{S}$-category*. For instance, if $\mathcal{C} = \mathcal{CW}$, the category of CW-complexes, then $\pi_0\mathcal{CW} = Ho(CW)$, the corresponding homotopy category. Similarly we could obtain a groupoid enriched category using the fundamental groupoid (cf. Gabriel and Zisman, [64]), that is, by applying the fundamental groupoid functor, Π_1, to each 'hom'

$$(\Pi_1\mathcal{C})(X,Y) = \Pi_1(\mathcal{C}(X,Y)).$$

This works because Π_1 preserves products. (We will see many similar results later, in which the type of enriched structure is transformed using a 'monoidal functor', i.e., one that is compatible with the monoidal category structures being used. All will be revealed later, in Chapter **??**.)

Remarks: This notion of a groupoid enriched category has been called a *track category* by Baues; see [15], for instance. The terminology is not quite precise enough for our uses as we may have track n-categories to handle later on, so we will call this 2-dimensional version a *track 2-category*. Formally we have:

Definition: A 2-category, $\mathbf{C}$, is a *track 2-category* or a *groupoid enriched category* if each $\mathbf{C}(x,y)$ is a groupoid.

These track 2-categories / groupoid enriched categories have a reasonably rich 'abstract' homotopy theory, as is shown by the book by Gabriel and Zisman, [64], or the article by Fantham and Moore, [62]. More recently they have been used extensively by Baues, [15].

One can 'do' some elementary homotopy theory in *any* $\mathcal{S}$-category, $\mathcal{C}$, by saying that two maps $f_0, f_1 : X \to Y$ in $\mathcal{C}$ are *homotopic* if there is an $H \in \mathcal{C}(X,Y)_1$ with $d_0H = f_1$, $d_1H = f_0$.

This theory will not be very rich, however, unless at least some low dimensional Kan conditions are satisfied.

Definition: The $\mathcal{S}$-category, $\mathcal{C}$, is called *locally Kan* if each $\mathcal{C}(X,Y)$ is a Kan complex; *locally weakly Kan* if $\ldots$, etc.

(If you have not met 'weak Kan complexes', you will soon meet them in earnest! We will define them properly before using them, so don't worry.)

The theory is 'geometrically' nicer to work with if $\mathcal{C}$ is *tensored* or *cotensored*.

5.3.2 Tensoring and Cotensoring

We have already met the idea of tensoring and cotensoring briefly when discussing simplicial homotopies, (page **??** in section **??**). The notions of tensors and cotensors make sense in any enriched category setting, but here we will just handle the case of simplicially enriched category.

Definition: If for all $K \in \mathcal{S}$, $X, Y, \in \mathcal{C}$, there is an object $K\bar{\otimes}X$ in $\mathcal{C}$ such that

$$\mathcal{C}(K\bar{\otimes}X, Y) \cong \mathcal{S}(K, \mathcal{C}(X, Y)$$

naturally in K, X and Y, then $\mathcal{C}$ is said to be *tensored* over $\mathcal{S}$.

Definition: Dually, if we require objects $\bar{\mathcal{C}}(K, Y)$ such that

$$\mathcal{C}(X, \bar{\mathcal{C}}(K, Y)) \cong \mathcal{S}(K, \mathcal{C}(X, Y)$$

then we say $\mathcal{C}$ is *cotensored* over $\mathcal{S}$.

Remark on terminology: In many ways this terminology is not a good one. Usually 'tensors' are given by colimit type constructions, whilst cotensors are limit-type constructions. A cotensor is interpreted as if it was a function or mapping 'space', and in the simple case of a *Set*-enriched setting, (i.e., standard category theory) is a *power* operation. If X, Y are objects in a category $\mathcal{C}$ and K is just a set, $\bar{\mathcal{C}}(K, Y)$ is Y^K, the K-fold power of Y, that is, the product of K-many copies of the object, Y. Dually $K\bar{\otimes}X$ will be the K-fold copower of X, that is, the coproduct of K-many copies of the object X. Because of this, an alternative terminology to the above has been suggested:

'standard'	alternative
tensored	copowered
cotensored	powered

(see the discussion of this in the nLab, [105].) (This terminology is probably still unstable but should stabilise soon.)

The example that we have seen most of this type of structure is in $\mathcal{S}$, where, for K in $\mathcal{S}$, and, this time, also X in $\mathcal{S}$, $K\bar{\otimes}X$ is just $K \times X$ and, dually, for Y in $\mathcal{S}$, $\bar{\mathcal{C}}(K, Y)$ is $\underline{S}(K, Y)$, the simplicial function space of maps from K to Y. To gain some intuitive feeling for these two concepts in general, we can think of $K\bar{\otimes}X$ as being 'K product with X', and $\bar{\mathcal{C}}(K, Y)$ as the object of functions from K to Y. These words do not, as such, make sense in all generality, but do tell one the sort of tasks these constructions will be set to do. They will not be much used explicitly here, however, their application to constructing homotopy limits and colimits will be looked at in detail later on.

The following also gives an indication of other uses. Some of the terminology has not been explicitly explained, but the results do give an idea of the structure available.

Proposition 36 *(cf. Kamps and Porter, [81]) If $\mathcal{C}$ is a locally Kan $\mathcal{S}$-category tensored over $\mathcal{S}$, then, taking $I \times X := \Delta[1]\bar{\otimes}X$, we get a good cylinder functor such that for the cofibrations relative to I and weak equivalences taken to be homotopy equivalences, the category $\mathcal{C}$ has a cofibration category structure.* ■

A cofibration category structure is just one of many variants of the abstract homotopy theory structure introduced to be able to push through homotopy type arguments in particular settings. There are variants of this result, due to Kamps, see references in [81], where $\mathcal{C}$ is both tensored and cotensored over $\mathcal{S}$ and the conclusion is that $\mathcal{C}$ has a Quillen model category structure. The examples of locally Kan $\mathcal{S}$-categories include Top, and Kan, that is the full subcategory of $\mathcal{S}$ given by the Kan complexes, also *Grpd* and Crs, but not Cat or $\mathcal{S}$ itself.

5.4 Nerves and Homotopy Coherent Nerves

Before we get going on this section, it will be a good idea to bring to the fore, as promised, the definitions of *weak Kan complex* (or *quasi-category*). We first recall and repeat from the first chapter, the notions of Kan fibration and Kan complex, as these are central to what follows and it is convenient not to have to be flipping back and fore to the earlier discussion.

5.4.1 Kan and weak Kan complexes

As usual, we set $\Delta[n] = \mathbf{\Delta}(-,[n]) \in \mathcal{S}$, then for each i, $0 \leq i \leq n$, we can form a subsimplicial set, $\Lambda^i[n]$, of $\Delta[n]$ by discarding the top dimensional n-simplex (given by the identity map on $[n]$) and its i^{th} face. We must also discard all the degeneracies of these simplices. This informal definition does not give a 'picture' of what we have, so we will list the various cases for $n = 2$.

$$\Lambda^0[2] = \begin{tikzcd} & 1 \arrow[dr, dotted, dash, "\leftarrow 0^{th}\text{ face missing}"] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$$\Lambda^1[2] = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr, dotted, dash] & & 2 \end{tikzcd}$$

$$\Lambda^2[2] = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur, dotted, dash] \arrow[rr] & & 2 \end{tikzcd}$$

A map $p : E \to B$ is a *Kan fibration* if given any n, i, as above, and any (n,i)-horn in E, i.e., any map $f_1 : \Lambda^i[n] \to E$, and n-simplex, $f_0 : \Delta[n] \to B$, such that

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] & B \end{tikzcd}$$

commutes, then there is an $f : \Delta[n] \to E$ such that $pf = f_0$ and $f.inc = f_1$, i.e., f lifts f_0 and extends f_1.

A simplicial set, K, is a *Kan complex* if the unique map $K \to \Delta[0]$ is a Kan fibration. This is equivalent to saying that every horn in K has a filler, i.e., any $f_1 : \Lambda^i[n] \to K$ extends to an

$f : \Delta[n] \to K$. This condition looks to be purely of a geometric nature but in fact has an important algebraic flavour; for instance, if $f_1 : \Lambda^1[2] \to K$ is a horn, it consists of a diagram

$$\begin{tikzcd} & {} \arrow[dr, "b"] & \\ {} \arrow[ur, "a"] & & {} \end{tikzcd}$$

of 'composable' arrows in K. If f is a filler, it looks like

$$\begin{tikzcd} & {} \arrow[dr, "b"] \arrow[d, phantom, "f"] & \\ {} \arrow[ur, "a"] \arrow[rr, "c"'] & {} & {} \end{tikzcd}$$

and one can think of the third face c as a composite of a and b. This 'composite' c is not usually uniquely defined by a and b, but is determined 'up to homotopy'. If we write $c = ab$ as a shorthand then if $g_1 : \Lambda^0[2] \to K$ is a horn, we think of g_1 as being

$$\begin{tikzcd} & {} & \\ {} \arrow[ur, "d"] \arrow[rr, "e"'] & & {} \end{tikzcd}$$

and to find a filler is to find a diagram

$$\begin{tikzcd} & {} \arrow[dr, dashed, "x"] & \\ {} \arrow[ur, "d"] \arrow[rr, "e"'] & & {} \end{tikzcd}$$

and thus to 'solve' the equation $dx = e$ for x in terms of d and e. It thus requires, in general, some approximate inverse for d, in fact, taking e to be a degenerate 1-simplex puts one in exactly such a position. Thus Kan complexes have a very weak 'algebraic' structure. There is a sort of composition, *up to homotopy*, which is sort of associative, *up to homotopy*, and has sort of inverses, yes, you guessed, *up to homotopy*.

In many useful cases, we do not always have inverses and so want to discard any requirement that would imply they always exist. This leads to the weaker form of the Kan condition in which in each dimension no requirement is made for the existence of fillers on horns that miss out the zeroth or last faces. More exactly:

Definition: A simplicial set $\mathbf{K}$ is a *weak Kan complex* or *quasi-category* if for any n and $0 < k < n$, any (n, k)-horn in K has a filler.

Remark: Joyal, [80], uses the term *inner horn* for any (n, k)-horn in K with $0 < k < n$. The two remaining cases are then conveniently called *outer horns*.

5.4.2 Categorical nerves

As we saw in section 1.3.1, the categorical analogue of the singular complex is the nerve: if C is a category, its *nerve*, $Ner(\mathsf{C})$, is the simplicial set with $Ner(\mathsf{C})_n = Cat([n], \mathsf{C})$, where $[n]$ is the category associated to the finite ordinal $[n] = \{0 < 1 < \ldots < n\}$. The face and degeneracy maps are the obvious ones using the composition and identities in C.

The following is well known and easy to prove (i.e., **left to you**).

Lemma 28 *(i) $Ner(\mathsf{C})$ is always weakly Kan.*
(ii) $Ner(\mathsf{C})$ is Kan if and only if C is a groupoid. ∎

Of course more is true. Not only does any inner horn in $Ner(\mathsf{C})$ have a filler, it has exactly one filler. To express this with maximum force, the following idea, often attributed to Graeme Segal or to Grothendieck, is very useful.

Let $p > 0$, and consider the increasing maps, $e_i : [1] \to [p]$, given by $e_i(0) = i$ and $e_i(1) = i + 1$. For any simplicial set, A, considered as a functor $A : \mathbf{\Delta}^{op} \to Sets$, we can evaluate A on these e_i and, noting that $e_i(1) = e_{i+1}(0)$, we get a family of functions $A_p \to A_1$, which yield a cone diagram, for instance, for $p = 3$:

$$\begin{tikzcd}
A_3 \arrow[rrrd, "A(e_1)"] \arrow[rrdd, "A(e_2)"] \arrow[rddd, "A(e_3)"'] & & & \\
& & & A_1 \arrow[d, "d_0"] \\
& & A_1 \arrow[r, "d_1"] \arrow[d, "d_0"] & A_0 \\
& A_1 \arrow[r, "d_1"'] & A_0 &
\end{tikzcd}$$

and in general, thus yield a map

$$\delta[p] : A_p \to A_1 \times_{A_0} A_1 \times_{A_0} \ldots \times_{A_0} A_1.$$

The maps, $\delta[p]$, have been called the *Segal maps*.

Lemma 29 *If $A = Ner(\mathsf{C})$ for some small category C, then for A, the Segal maps are bijections.*

Proof: A simplex $\sigma \in Ner(\mathsf{C})_p$ corresponds uniquely to a composable p-chain of arrows in C, and hence exactly to its image under the relevant Segal map. ∎

Better than this is true:

Proposition 37 *If A is a simplicial set such that the Segal maps are bijections, then there is a category structure on the directed graph,*

$$\begin{tikzcd}
A_1 \arrow[r, shift left=2] \arrow[r] & A_0 \arrow[l, shift left=2]
\end{tikzcd} ,$$

making it a category whose nerve is isomorphic to the given A.

Proof: To get composition you use

$$A_1 \times_{A_0} A_1 \overset{\cong}{\to} A_2 \overset{d_1}{\to} A_1.$$

Associativity is given by A_3. The other laws are easy, and illuminating, to check. ∎

The condition 'Segal maps are a bijection' is closely related to notions of 'thinness' as used by Brown and Higgins in the study of crossed complexes and their relationship to ω-groupoids, (see, for instance, [31], and here in our discussion of T-complexes, starting on page 30), and it also relates to Duskin's 'hypergroupoid' condition, [56].

Another result that is sometimes useful is a refinement of the 'groupoids give Kan complexes' lemma, Lemma 1 on page 29. The proof is 'the same' and is equally **left to the reader**.

Lemma 30 *Let $A = Ner(\mathsf{C})$, the nerve of a category C.*

(i) Any $(n, 0)$-horn

$$f : \Lambda^0[n] \to A$$

for which $f(01)$ is an isomorphism has a filler. Similarly any (n, n)-horn $g : \Lambda^n[n] \to A$ for which $g(n-1\ n)$ is an isomorphism, has a filler.

(ii) Suppose f is a morphism in C with the property that, for any n, any $(n, 0)$-horn $\varphi : \Lambda^0[n] \to A$ having f in the $(0, 1)$ position, has a filler, then f is an isomorphism. (Similarly with $(n, 0)$ replaced by (n, n) with the obvious changes.) ■

Again the proof is not hard and reveals some neat arguments, so ... .

Remark: Joyal in [80] suggested that the name 'weak Kan complex', as introduced by Boardman and Vogt, [20], could be changed to that of 'quasi-category' to stress the analogy with categories *per se* as '*Most concepts and results of category theory can be extended to quasi-categories*', [80].

It would have been nice to have explored Joyal's work on quasi-categories more fully, e.g. [80], but that would take us too far from our central themes. The following few sections just skate the surface of that theory.

5.4.3 Quasi-categories

Categories yield quasi-categories via the nerve construction as we have seen. Quasi-categories yield categories by a 'fundamental category' construction that is left adjoint to nerve. This can be constructed using the free category generated by the 1-skeleton of A, and then factoring out by a congruence generated by the basic relations : $gf \equiv h$, one for each commuting 1-sphere (g, h, f) in A. By a *1-sphere* is meant a map $a : \partial\Delta[2] \to A$, thus giving three faces, (a_0, a_1, a_2), linked in the obvious way. The 1-sphere is said to be *commuting* if there is a 2-simplex, $b \in A_2$, such that $a_i = d_i b$ for $i = 0, 1, 2$.

Definition: The *fundamental category of a quasi-category*, A, is the category with presentation:

• generators = the 1-skeleton of A,

and

• relations $gf \equiv h$ as above.

This 'fundamental category' functor also has a very neat description due to Boardman and Vogt. (The treatment here is, again, adapted from [80].)

We assume given a quasi-category, A. Write $gf \sim h$ if (g, h, f) is a commuting 1-sphere. Let $x, y \in A_0$ and let $A_1(x, y) = \{f \in A_1 \mid x = d_1 f, y = d_0 f\}$. If $f, g \in A_1(x, y)$, then, suggestively writing $s_0 x = 1_x$,

Lemma 31 *The four relations $f1_x \sim g$, $g1_x \sim f$, $1_y f \sim g$ and $1_y g \sim f$ are equivalent.* ■

The proof is easy and is **left as an exercise**.

We will say $f \simeq g$ if any of these is satisfied and call $\simeq$, the *homotopy relation*. It is an equivalence relation on $A_1(x, y)$. Set $ho\, A_1(x, y) = A_1(x, y)/\simeq$.

If $f \in A_1(x, y)$, $g \in A_1(y, z)$ and $h \in A_1(x, z)$, then the relation $gf \sim h$ induces a map:

$$ho\, A_1(x, y) \times ho\, A_1(y, z) \to ho\, A_1(x, z).$$

Proposition 38 *The maps*

$$ho\, A_1(x, y) \times ho\, A_1(y, z) \to ho\, A_1(x, z)$$

give a composition law for a category, $ho\, A$, the homotopy category of A. ■

Definition: This category, $ho\, A$, is called the *homotopy category* of A.

Of course, $ho\, A$ is the fundamental category of A up to natural isomorphism. From previous comments we have:

Corollary 9 *A quasi-category A is a Kan complex if and only if $ho\, A$ is a groupoid.* ■

5.4.4 Homotopy coherent diagrams and homotopy coherent nerves

(The notion was explicitly introduced by Cordier, [42], adapting ideas from Boardman and Vogt, [20]. There is an overview of this theory in [109] and a thorough introduction in [81]. The construction of the homotopy coherent nerve is also used, extensively, by Lurie in [91], and by Hinich, [74].)

Before handling this topic, we quickly recall some of the intuition behind homotopy coherent (h. c.) diagrams, as we saw a few pages back.

A diagram indexed by the small category, [2],

$$\begin{tikzcd} & X(1) \arrow[dr, "X(12)"] & \\ X(0) \arrow[ur, "X(01)"] \arrow[rr, "X(02)"'] & & X(2) \end{tikzcd}$$

is h. c. if there is specified a homotopy

$$X(012) : I \times X(0) \to X(2),$$

$$X(012) : X(02) \simeq X(12)X(01).$$

For a diagram indexed by [3]: Draw a 3-simplex, marking the vertices $X(0), \ldots, X(3)$, the edges $X(ij)$, etc., the faces $X(ijk)$, etc. The homotopies $X(ijk)$ fit together to make the sides of a square

$$\begin{tikzcd}
X(13)X(01) \arrow[r, "X(123)X(01)"] & X(23)X(12)X(01) \\
X(03) \arrow[u, "X(013)"] \arrow[r, "X(023)"'] & X(23)X(02) \arrow[u, "X(23)X(012)"']
\end{tikzcd}$$

and the diagram is made h. c. by specifying a second level homotopy

$$X(0123) : I^2 \times X(0) \to X(3)$$

filling this square.

These can be continued for larger $[n]$, as we have hinted.

We have seen that the 'same' diagrams occur when we draw what seems to be a reasonable example of the intuitive form of homotopy coherent diagram in Top and in the $\mathcal{S}$-categories, $S(\mathbb{A})$. This suggests the definition of a homotopy coherent diagram in an arbitrary $\mathcal{S}$-category. This form is due to Cordier, [42], extending the earlier work of Boardman and Vogt.

Definition: Let $\mathbb{A}$ be a small category and $\mathcal{B}$, an $\mathcal{S}$-category.

(i) A *homotopy coherent diagram of type* $\mathbb{A}$ *in* $\mathcal{B}$ is a $\mathcal{S}$-functor $F : S(\mathbb{A}) \to \mathcal{B}$.

(ii) If $F_0, F_1 : S(\mathbb{A}) \to \mathcal{B}$ are two such diagrams, a *homotopy coherent map* between them is a diagram of type $\mathbb{A} \times [1]$ agreeing with F_0 on $\mathbb{A} \times \{0\}$ and with F_1 on $\mathbb{A} \times \{1\}$.

Of course, we refer to $\mathbb{A}$ as the *template* of the h.c. diagram, F.

We should pause to examine this notion of homotopy coherent map in more detail, via our low dimensional examples, i.e., with $\mathbb{A} = [n]$ for small values of n.

For $n = 0$, this is unenlightening: $F_0, F_1 : S[0] \to \mathcal{B}$, so they are really just two objects of $\mathcal{B}$, and a h.c. map between them in then just a map between $F_0(0)$ and $F_1(0)$ in $\mathcal{B}$.

For $n = 1$, it is already a much richer picture. This time, F_0 and F_1 pick out two maps in $\mathcal{B}$, namely $F_i(0) \stackrel{F_i(01)}{\to} F_i(1)$ for $i = 0, 1$. A homotopy coherent map $\eta : F_0 \to F_1$ is a h.c. diagram of type $[1] \times [1]$, so is a square of form

$$\begin{tikzcd}
F_1(0) \arrow[r] & F_1(1) \\
F_0(0) \arrow[u] \arrow[r] \arrow[ur] \arrow[ur, phantom, shift left=3ex, "\Nwarrow"] \arrow[ur, phantom, shift right=3ex, "\Searrow"] & F_0(1) \arrow[u]
\end{tikzcd}$$

and will specify $\eta(i) : F_0(i) \to F_1(i)$ for $i = 0, 1$, but also a diagonal map, which we will write $\eta_0^1 : F_0(0) \to F_1(1)$, then also we will have homotopies as shown from η_0^1 to $F_1(01)\eta(0)$ and to $\eta(1)F_0(01)$, respectively.

It is worthwhile pausing to note that, in this simplicial approach, there is an avoidance of questions of directions of 2-cells (and higher order ones). Often when looking at diagrams showing lax or pseudo morphisms between lax or pseudo functors, one or other of the directions is chosen, e.g., here it might typically be $\eta : \eta(1)F_0(01) \Rightarrow F_1(01)\eta(0)$. If we are in a 'pseudo' context, this choice, although arbitrary, is somewhat immaterial as η will be invertible, but this need not be the case for a lax morphism. Nothing dictates which direction is 'better' and both are present in this simplicial approach. If someone gives you $\eta : \eta(1)F_0(01) \Rightarrow F_1(01)\eta(0)$, you can take $\eta_0^1 = \eta(1)F_0(01)$ and set the bottom right homotopy to be the identity. Likewise if η is presented in the reverse direction, just set the top left cell to be the identity two cell and use the given η in the bottom right. Some people do not like this as they prefer one choice or other, usually for a good reason from the situation being handled, yet, simplicially, it is more or less required to have the diagonal and the two 2-cells.

For $n = 2$, we have a prism, $[2] \times [1]$, and you have to specify η on three tetrahedra in this, agreeing on the overlaps. Here is a possible notation and the beginnings of a detailed discussion which can be extended to higher dimensions. (The rest is not hard, but does **really involve reader participation**!)

$$\begin{tikzcd}
& (1,0) \arrow[rrr] \arrow[ddr] \arrow[ddrrrr] & & & (1,1) \arrow[ddr] & \\
(0,0) \arrow[ur] \arrow[rrr] \arrow[drr] & & & (0,1) \arrow[ur] \arrow[drr] & & \\
& & (2,0) \arrow[rrr] & & & (2,1)
\end{tikzcd}$$

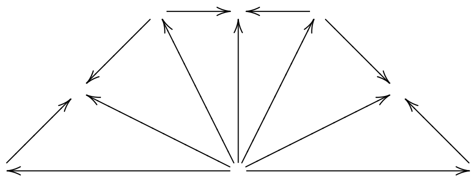
We suggest a matrix notation. For this the use of column 'vectors' is preferable to rows, so (1,0) becomes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a vertex label; the edge from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is then clearly $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; the shown diagonal is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. (Two diagonals have been left out of the diagram.)

We mentioned three tetrahedra. These are

$$\sigma_0 = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

The first and second share a d_2-face, namely $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, whilst the second and third share a d_1-face, i.e., $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$.

The comments above about 'orientation' or 'direction' are even more pertinent here. For each tetrahedron, we have a copy of $S[3]$, so in particular $S[3](0,3)$ is there 3 times. As $S[3](0,3)$ is a square, $\Delta[1]^2$, we have 6 two simplices in $S([2] \times [1])((0,0),(2,1))$. They fit together to give half a hexagon:



Each subdivided segment is a square in disguise! (You get half a hexagon because the prism is half of the cube $[1]^3$, and $S([1]^3)$ is a barycentrically subdivided hexagon.) Of the six 2-simplices, if you check their orientation half go anticlockwise, half go clockwise. Later in our discussion of 2-dimensional descent data, we will have a prismical diagram. In each rectangular side face, we choose the convention as above, putting the 'active' face in one of the two 2-simplices. This means three of the boundary arrows in the above will be set to be equalities. The diagram we will use there is a commuting pentagon of 2-cells in a 2-category, and, of course, this can be derived from the above by noting that in 2-categories, there are no 3-cells, so $S([2] \times [1])((0,0),(2,1))$ will be mapped to a category, something like $\mathcal{B}(F_0(0), F_1(2))$, but that has no non-identity 2-cells, so the 2-simplices will be sent to identity homotopies. The other input is that $5 = 8 - 3$ (proof left to the reader - no calculators permitted - other than your fingers!!!) We will refer back to this when we are looking at 2-dimensional descent. It permits us to see the phenomena there as being very much akin to those with homotopy coherence.

This type of combinatorial analysis can be very useful when handling maps of homotopy coherent diagrams and relating them to other descriptions (lax, pseudo , etc.) of the same situations. It is not the only way of handling these ideas however, and the simplicial set of maps between two $\mathcal{S}$-functors, $F, G : S(\mathbb{A}) \to \mathcal{B}$, can be handled categorically as well. The basic intuition is, however, very much the same, and the resulting problems are there, whichever way you approach this. Use of more high powered categorical machinery, quasi-categories, etc. can make the theory much easier to apply, but also then you need to keep in sight the basic intuitions and to see how the combinatorics related to that is encoded in the machine you are using.

We mentioned 'problems' ... what are they?

In general, homotopy coherent maps, as defined here, need not compose, even when they might be expected to. The problem is analogous to that of composing homotopies between simplicial maps, that we met a short while ago. Unless the codomains are Kan complexes, there is no guarantee that such homotopies can be composed. Even when they compose, of course, there will, in general, be many composites. Those composites *will* be themselves homotopic and so on. Here with homotopy coherent maps, provided that the ambient category, $\mathcal{B}$, is *locally weakly Kan*, (i.e., is 'quasi-category'-enriched), then they do compose, up to homotopy. The result is a sort of 'A_∞-category' structure, (see Batanin's paper, [13]), but also has a quasi-categorical description, which we will meet shortly. One can also use Verity's theory of complicial sets, [128] and their weakened form, [129–131]. These are closely related to the simplicial T-complexes we saw in section 1.3.6.

The theory of homotopy coherence was initially developed explicitly by Vogt, [132], following methods introduced with Boardman, [20], (see also the references in that source for other earlier papers on the area), then Cordier, [42], provided the simple $\mathcal{S}$-category theory way of working with h. c. diagrams and hence released an 'arsenal' of categorical tools for working with h. c. diagrams. Some of that is worked out in the papers, [44–47]. We illustrate this with some results taken from those sources.

(i) If $X : \mathbb{A} \to Top$ is a commutative diagram and we replace some of the $X(a)$ by homotopy equivalent $Y(a)$ with specified homotopy equivalence data:

$$f(a) : X(a) \to Y(a), \quad g(a) : Y(a) \to X(a)$$

$$H(a) : g(a)f(a) \simeq Id, \quad K(a) : f(a)g(a) \simeq Id,$$

then we can combine these data into the construction of a h. c. diagram Y based on the objects

$Y(a)$ and homotopy coherent maps

$$f : X \to Y, \quad g : Y \to X, \text{ etc.,}$$

making X and Y homotopy equivalent as h. c. diagrams. In other words, our earlier simple examples can be handled for any indexing category. (This is 'really' a result about quasi-categories, see [80].)

(ii) Vogt, [132]. If $\mathbb{A}$ is a small category, there is a category $Coh(\mathbb{A}, Top)$ of h. c. diagrams and homotopy classes of h. c. maps between them. Moreover there is an equivalence of categories

$$Coh(\mathbb{A}, Top) \xrightarrow{\simeq} Ho(Top^{\mathbb{A}})$$

where $Ho(Top^{\mathbb{A}})$ is obtained from the category of functors from $\mathbb{A}$ to Top bu inverting objectwise homotopy equivalences.

This was extended replacing Top by a general locally Kan simplicially enriched complete category, $\mathcal{B}$, in [43].

(iii) Cordier (1980), [42]. Given $\mathbb{A}$, a small category, then the $\mathcal{S}$-category $S(\mathbb{A})$ is such that a h. c. diagram of type $\mathbb{A}$ in Top is given precisely by an $\mathcal{S}$-functor

$$F : S(\mathbb{A}) \to Top$$

This suggested the extension of h. c. diagrams to other contexts such as a general locally Kan $\mathcal{S}$-category, $\mathcal{B}$, and suggests the definition of homotopy coherent diagram in a $\mathcal{S}$-category and thus a h. c. nerve of an $\mathcal{S}$-category.

Definition: (Cordier (1980), [42]) Given a simplicially enriched category $\mathcal{B}$, the *homotopy coherent nerve* of $\mathcal{B}$, denoted $Ner_{h.c.}(\mathcal{B})$, is the simplicial 'set' with

$$Ner_{h.c.}(\mathcal{B})_n = \mathcal{S}{-}Cat(S[n], \mathcal{B}),$$

and with the induced face and degeneracy maps.

Remark on terminology: Cordier, [42], initially used the term 'homotopy coherent nerve' for the above as he was primarily interested in its use in that area although in his subsequent work with Porter, [44–47], the quasi-categorical and ∞-categorical aspects were often a priority. Lurie, [91], has called this the *simplicial nerve functor* as his applications are not explicitly concerned with homotopy coherence.

To understand simple h. c. diagrams and thus $Ner_{h.c.}(\mathcal{B})$, we will unpack the definition of homotopy coherence.

The first thing to note is that, as we saw, for any n and $0 \leq i < j \leq n$, $S[n](i, j) \cong \Delta[1]^{j-i-1}$, the $(j - i - 1)$-cube given by the product of $j - i - 1$ copies of $\Delta[1]$. Thus we can reduce the higher homotopy data to being just that, maps from higher dimensional cubes.

Next some notation:

Given simplicial maps

$$f_1 : K_1 \to \mathcal{B}(x, y),$$

$$f_2 : K_2 \to \mathcal{B}(y, z),$$

we will denote the composite

$$K_1 \times K_2 \to \mathcal{B}(x, y) \times \mathcal{B}(y, z) \xrightarrow{c} \mathcal{B}(x, z)$$

just by $f_2.f_1$ or $f_2 f_1$. (We already have seen this in the h. c. diagram above for $\mathbb{A} = [3]$. $X(123)X(01)$ is actually $X(123)(I \times X(01))$, whilst $X(23)X(012)$ is exactly what it states.)

Suppose now that we have the h. c. diagram, $F : S(\mathbb{A}) \to \mathcal{B}$. This is an $\mathcal{S}$-functor and so:

- to each object a of $\mathbb{A}$, it assigns an object $F(a)$ of $\mathcal{B}$;
- to each string of composable maps in $\mathbb{A}$,

  $$\sigma = (f_0, \ldots, f_n)$$

  starting at a and ending at b, it assigns a simplicial map

  $$F(\sigma) : S(\mathbb{A})(0, n+1) \to \mathcal{B}(F(a), F(b)),$$

  that is, a higher homotopy

  $$F(\sigma) : \Delta[1]^n \to \mathcal{B}(F(a), F(b)),$$

  such that
- if $f_0 = id$, $F(\sigma) = F(\partial_0\sigma)(proj \times \Delta[1]^{n-1})$;
- if $f_i = id$, $0 < i < n$,

  $$F(\sigma) = F(\partial_i\sigma).(I^i \times m \times I^{n-i}),$$

  where $m : I^2 \to I$ is the multiplicative structure on $I = \Delta[1]$ induced by the 'max' function on $\{0, 1\}$;
- if $f_n = id$, $F(\sigma) = F(\partial_n\sigma)$;
- $_i\ F(\sigma)|(I^{i-1} \times \{0\} \times I^{n-i}) = F(\partial_i\sigma), 1 \leq i \leq n-1$;
- $_i\ F(\sigma)|(I^{i-1} \times \{1\} \times I^{n-i}) = F(\sigma_i').F(\sigma_i)$, where $\sigma_i = (f_0, \ldots, f_{i-1})$ and $\sigma' = (f_i, \ldots, f_n)$.

We have used ∂_i here for the face operators in the nerve of $\mathbb{A}$.

The specification of such a homotopy coherent diagram can be split into two parts:

(a) specification of certain homotopy coherent *simplices*, i.e., elements in $Ner_{h.c.}(\mathcal{B})$;
and

(b) specification, via a simplicial mapping from $Ner(\mathbb{A})$ to $Ner_{h.c.}(\mathcal{B})$, of how these individual parts (from (a)) of the diagram are glued together.

The second part of this is easy as it amounts to a simplicial map $Ner(\mathbb{A}) \to Ner_{h.c.}(\mathcal{B})$, and so we are left with the first part. The following theorem was proved by Cordier and Porter, [43], but many of the ideas for the proof were already in Boardman and Vogt's lecture notes, like so much else!

Theorem 10 *([43]) If $\mathcal{B}$ is a locally Kan $\mathcal{S}$-category, then $Ner_{h.c.}(\mathcal{B})$ is a quasi-category.* ■

It is not clear what happens if $\mathcal{B}$ is only locally weakly Kan, is $Ner_{h.c.}(\mathcal{B})$ then a quasi-category? It is probably a known result, maybe even clear, but may not be in published form.

The proof of the theorem is in the paper, [43], and is not too complex. The essential feature is that the very definition (unpacked version) of homotopy coherent diagram makes it clear that parts of the data have to be composed together, (recall the composition of simplicial maps

$$f_1 : K_1 \to \mathcal{B}(x, y),$$

$$f_2 : K_2 \to \mathcal{B}(y, z),$$

above and how important that was in the unpacked definition).

We thus have that a homotopy coherent diagram 'is' a simplicial map, $F : Ner(\mathbb{A}) \to Ner_{h.c.}(\mathcal{B})$, and that $Ner_{h.c.}(\mathcal{B})$ is a quasi-category. Of course, the usual proof that, if X and Y are simplicial sets, and Y is Kan, then $\underline{\mathcal{S}}(X, Y)$ is Kan as well, extends to having Y a quasi-category and the result being a quasi-category. Earlier we referred to $Coh(\mathbb{A}, \mathcal{B})$ in connection with Vogt's theorem. The neat way of introducing this is as $ho\,\mathcal{S}(Ner(\mathbb{A}), Ner_{h.c.}(\mathcal{B}))$, the fundamental category of the function quasi-category. In fact, this is essentially the way that Vogt first described it.

If $\mathcal{A}$ and $\mathcal{B}$ are both $\mathcal{S}$-categories, and $F : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{S}$-functor, then clearly F induces a simplicial map

$$Ner_{h.c.}(F) : Ner_{h.c.}(\mathcal{A}) \to Ner_{h.c.}(\mathcal{B}).$$

In other words $Ner_{h.c.}$ is a functor from $\mathcal{S}-Cat$ to $\mathcal{S}$, ignoring any problems due to 'size' of the categories involved. We will see later (Proposition 42 and the discussion around that result) that there may be simplicial maps between $Ner_{h.c.}(\mathcal{A})$ and $Ner_{h.c.}(\mathcal{B})$ that do not come from $\mathcal{S}$-functors.

As the category, $\mathcal{S}-Cat$, of (small) $\mathcal{S}$-categories and $\mathcal{S}$-functors between them is cocomplete, there is a left adjoint to this nerve functor in the usual way. The general picture of such adjoint pairs induced by some 'models' here looks like this: we have $S : \mathbf{\Delta} \to \mathcal{S}-Cat$ and $\Delta : \mathbf{\Delta} \to \mathcal{S}$, the Yoneda embedding, and these induce the nerve and 'realisation' adjoint pair. (If you replace $\mathcal{S}-Cat$ by Top you get the singular complex / geometric realisation adjoint pair, that you have met earlier.) As the nerve functor has a left adjoint, it preserves limits and, in particular, products.

5.4.5 Simplicial coherence and models for homotopy types

Before we look at more direct applications of simplicially based homotopy coherence, there is a point that is worth noting for the links with algebraic and categorical models for homotopy types. The $\mathcal{S}$-categories, $S[n]$, contain a lot of the information needed for the construction of such models. A good example of this is the interchange law and its links with Gray categories and Gray groupoids.

Consider $S[4]$. The important information is in the simplicial set $S[4](0,4)$. This is a 3-cube, so is still reasonably easy to visualise. Here it is. The notation is not intended to be completely consistent with earlier uses, but is meant to be more or less self explanatory.

$$\begin{tikzcd}
 & (01)(13)(34) \arrow[rr] & & (01)(12)(23)(34) \\
(01)(14) \arrow[ur] \arrow[rr] & & (01)(12)(24) \arrow[ur] & \\
 & (03)(34) \arrow[uu] \arrow[rr] & & (02)(23)(34) \arrow[uu] \\
(04) \arrow[uu] \arrow[ur] \arrow[rr] & & (02)(24) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

This looks mysterious! A 4-simplex has 5 vertices, and hence 5 tetrahedral faces. Each of the 5 tetrahedral faces will contribute a square to the above diagram, yet a cube has 6 square faces! Where does the 'extra' face come from? (Things get 'worse' in $S[5](0,5)$, which is a 4-cube, so has 8 cubes as its faces, but $\Delta[5]$ has only 6 faces.) Back to the 'extra' face, this is

$$\begin{tikzcd}
(01)(12)(24) \arrow[r, "(01)(12)(234)"] & (01)(12)(23)(34) \\
(02)(24) \arrow[u, "(012)(24)"] \arrow[r, "(02)(234)"'] \arrow[ur, phantom, "(012)(234)"] & (02)(23)(34). \arrow[u, "(012)(23)(34)"']
\end{tikzcd}$$

The arrow $(012) : (02) \to (01)(12)$ will, in a homotopy coherent diagram, make its appearence as the homotopy,

$$X(012) : I \times X(0) \to X(2),$$

$$X(012) : X(02) \simeq X(12)X(01),$$

thus this square implies that the homotopies $X(012)$ and $X(234)$ interact minimally. Drawing homotopies as 2-cells in the usual way, the square we have above is the interchange square and the interchange law will hold in this system provided this square is, in some sense, commutative. In models for homotopy n-types for $n \geq 3$, these interchange squares give part of the pairing structure between different levels of the model. They are there in, say, the Conduché model (2-crossed modules, cf. Conduché, [41] and here, section **??**) as the *Peiffer lifting*, and in the Loday model, (crossed squares, cf. [90]), as the h-map. In a general dimension, n, there will be pairings like this for any splitting of $\{0, 1, \ldots, n\}$ of the form $\{0.1. \ldots, k\}$ and $\{k, \ldots, n\}$. These are related to the Peiffer pairings that we have mentioned several times.

5.5 Useful examples

By the main title of this section, we intend to concentrate attention on the ways in which homotopy coherence techniques clarify what is going on at certain points of the development of cohomology and related areas. Mostly these are instances of more general results listed or mentioned earlier in this chapter.

5.5.1 G-spaces: discrete case

The first example concerns a G-space for G a discrete group. (For G a topological group, more subtle arguments are needed although, as we will see later, the basic idea is the same.) We therefore have a space, X, and an action

$$a : G \times X \to X, \qquad a(g, x) = g \cdot x,$$

being a continuous map from the product to X satisfying some rules. We have considered such a G-object in several different ways, and settings, not all of them 'spatial'. One was to consider the group, G, as a groupoid with a single object. This groupoid has usually been written $G[1]$, with the single object denoted by $*$ or similar. We then built a functor, $\mathbb{X} : G[1] \to Top$, as follows:

- $\mathbb{X}(*) = X$;
- if $g : * \to *$ in $G[1]$ and $x \in X$, then $\mathbb{X}(g) : \mathbb{X}(*) \to \mathbb{X}(*)$ is simply $\mathbb{X}(g)(x) = g \cdot x$, and, of course, $\mathbb{X}(g_1 g_2) = \mathbb{X}(g_1)\mathbb{X}(g_2)$.

If we need another description of functors than merely elementwise, (which can be awkward for categorification), it may help to replace the second part of the above by

$$\mathbb{X} : G[1](*, *) \to Top(\mathbb{X}(*), \mathbb{X}(*)),$$

as being the analogue of the usual : if $F : \mathbb{A} \to \mathbb{B}$, then, for any objects a_1, a_2 in $\mathbb{A}$, there is a map

$$F : \mathbb{A}(a_1, a_2) \to \mathbb{B}(F(a_1), F(a_2)),$$

which has to satisfy some composition preservation rules (and some tightening up on notation, since this F is really something like F_{a_1,a_2}, and so on).

The point of this second description is two fold. We have, once unpacked from its notation, just a function

$$G \to Top(X, X),$$

(and the codomain here is a monoid under composition of functions), which preserves multiplication and identity. The image of this function is thus within $Aut(X) \subseteq Top(X, X)$, the group of self homeomorphisms of X, and so we get back to the other description of an action as a homomorphism,

$$G \to Aut(X).$$

(If G is a topological group and Top is Cartesian closed, then $Aut(X)$ will be a topological group, and a continuous action will correspond to a *continuous* homomorphism of the same form. If G is a simplicial group and X is a simplicial set, we get back simplicial automorphisms and simplicial actions as we looked at earlier (in section **??**, starting on page **??**, and the section following that). Here, of course, $G[1]$ is a simplicially enriched groupoid and the action yields an $\mathcal{S}$-functor, $\mathbb{X} : G[1] \to \mathcal{S}$, and so on. (You should play around with the different types of contexts to see what works well and what less well.))

Each of these descriptions of G-actions is useful. Here we will take the description of a G-space as

$$\mathbb{X} : G[1] \to Top.$$

(From now on, we drop the 'blackboard' font, $\mathbb{X}$, for this and merely write X.) Now suppose that we replace our space X by a homotopy equivalent one, Y, (along a homotopy equivalence, ($f : X \to Y$, $f' : Y \to X$, $\mathbf{H}$ and $\mathbf{K}$)), then we do not usually get a G-action on Y. (The situation is, of course, essentially that which we examined in section 5.1, and it is worthwhile to see what a 'bare hands' approach gives in this situation.)

The theoretical, general, results that we have quoted give us a homotopy coherent diagram

$$Y : S(G[1]) \to \underline{Top},$$

where $\underline{Top}$ is the simplicial enrichment of Top.

Of course, there is nothing magical about $\underline{Top}$ here and we could have equally well have used $\mathcal{S}$ or a general simplicially enriched category, $\underline{\mathcal{B}}$. (Of course, for some purposes, we would need for $\mathcal{B}$ to be 'locally Kan' and / or for certain limits or colimits to exit, in order to get a neat theory here.)

The points to retain from this are that $S(G[1])$ is almost the 'free-group' comonadic simplicial resolution of G. It is a simplicial monoid, not a simplicial group however. We have *deformed* the group action to a homotopy coherent action and this is done by replacing G by a free simplicial resolution of G. (This is another instance of 'cofibrant replacement'.)The role of $Aut(X)$ is no longer viable. We cannot use $Aut(Y)$ in its place because, if we have $g \in G$, then we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "X(g)"'] & Y \arrow[d, dashed, "Y(\langle g\rangle)"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and $Y(\langle g\rangle) = fX(g)f'$, at least according to the recipe that we found in our earlier analysis. We cannot guarantee that $Y(\langle g\rangle)$ will be an 'automorphism' of Y. We do have $X(g^{-1}) : X \to X$, but then our algorithm for constructing Y gives $Y(\langle g^{-1}\rangle) = fX(g^{-1})f'$, so $Y(\langle g^{-1}\rangle)Y(\langle g\rangle) \simeq Y(\langle 1_G\rangle) \simeq 1_Y$. We thus do have $Y(\langle g\rangle)$ is a self equivalence (auto-equivalence) of Y, in our case, a self homotopy equivalence, but we could be in another context, e.g. Cat, and the same basic argument would work.

This is not the end of the example. We have

$$Y : S(G[1]) \to \mathcal{B},$$

but therefore we have a simplicial description of Y as

$$Y : Ner(G[1]) \to Ner_{h.c.}(\mathcal{B}).$$

We know what $Ner(G[1])$ is. It is what we have denoted BG, the classifying space of G. (Unlike the other contextfs where we have met it, however, it is the domain not the codomain of the relevant map.)

That gives us an additional intuition on several themes that we have met earlier, but there are others that are closely related where it is not so clear how it might help.

5.5.2 Lax and Op-lax functors and nerves for 2-categories

As we have mentioned lax functors several times informally, we should probably give a more formal definition, especially as the basic idea is clearly closely related to homotopy coherence in some 'intuitive' way.

Our earlier discussion, for instance in section **??**, related to a 'functor-like' mapping from a *category*, $\mathbb{A}$, into a 2-category, usually the 2-category Cat. We will give, below, a more general definition for when we have a 2-category, $\mathcal{A}$, as domain and a general 2-category, $\mathcal{B}$, as codomain for our generalised functor. To be able to relate back to the earlier case, it is useful to have some terminology to handle that situation.

Definition: Suppose $\mathcal{A}$ is a 2-category. We say that it is a *locally discrete* 2-category or is *locally 2-discrete* if, for each pair of objects, A_0, A_1 in $\mathcal{A}$, the category $\mathcal{A}(A_0, A_1)$ is a discrete category, (i.e., really just a set, so $\mathcal{A}$ has no non-identity 2-cells).

This will, thus, allow us to think of an ordinary category as being a 2-category, and it gives an embedding of Cat into $2-Cat$. We will shortly be considering a 2-category as an $\mathcal{S}$-category (as on page 146). We also will use such phrases as 'since $\mathbb{A}$ has no non-identity 2-cells' to indicate that we are considering $\mathbb{A}$ as a locally discrete 2-category, without making a fuss about it or denoting that version of $\mathbb{A}$ by some changed symbol. The natural tendency is then to extend this to saying that a 2-category, $\mathcal{A}$, 'has no non-identity 3-cells', although we have not considered 3-categories at all as yet.

If the 2-category is a locally discrete one, then, naturally, the resulting $\mathcal{S}$-category is a locally discrete $\mathcal{S}$-category, as well.

Suppose now that $\mathcal{A}$ and $\mathcal{B}$ are both 2-categories.

Definition: A *lax functor*, $\mathcal{F} = (F, c) : \mathcal{A} \to \mathcal{B}$, assigns

- to each object A of $\mathcal{A}$, an object, $F(A)$, of $\mathcal{B}$;
- to each pair of objects, A_0, A_1, of $\mathcal{A}$, a functor,
$$F : \mathcal{A}(A_0, A_1) \to \mathcal{B}(FA_0, FA_1);$$
- to each composable pair of 1-cells / morphisms, (f, g) of $\mathcal{A}$, a 2-cell,
$$c_{f,g} : F(g)F(f) \Rightarrow F(gf),$$
depending naturally on f and g, and to each object A of $\mathcal{A}$, a 2-cell, $c_A : id_{FA} \Rightarrow F(id_A)$,

such that the coherence conditions, below, are satisfied:

- for any composable triple, (f, g, h), of 1-cells / morphisms of $\mathcal{A}$, the diagram
$$\begin{tikzcd} F(hg)F(f) \arrow[r, Rightarrow, "c_{f,hg}"] & F(hgf) \\ F(h)F(g)F(f) \arrow[u, Rightarrow, "c_{g,h}\circ F(f)"] \arrow[r, Rightarrow, "F(h)\circ c_{f,g}"'] & F(h)F(gf) \arrow[u, Rightarrow, "c_{gf,h}"'] \end{tikzcd}$$
commutes;

- for any 1-cell, $f \in \mathcal{A}(A_0, A_1)$, the diagrams

$$\begin{tikzcd}
F(f)F(id_{A_0}) \arrow[r, Rightarrow, "c_{f,id_{A_0}}"] & F(f \circ id_{A_0}) = F(f) \arrow[dl, equal] \\
F(f) = F(f) \circ id_{F(A_0)} \arrow[u, Rightarrow, "F(f)c_{A_0}"] &
\end{tikzcd}$$

  and similarly for id_{A_1} on the other side, commute.

Remarks: (i) Of course, any 2-functor corresponds to a set of data as here, but with each $F(g)F(f) = F(gf)$ and all the $c_{f,g}$s being the relevant identities.

(ii) In some case, for each A, c_A is the identity map, i.e., the lax functor $\mathcal{F}$ preserves identities. In this case the terminology '*normal lax functor* is often used. This is consistent with the use of 'normalised' when referring to constructions such as the bar resolution. Most of the lax functors that we will meet will be 'normal'.

(iii) A quick look forward a few pages to page **??** and the definition of *(lax) monoidal functor* should convince you that the two ideas are closely related. Any 2-category is a 'strict' bicategory and any monoidal category 'is' a bicategory having just a single object, so bicategories (also called *weak 2-categories*) are a common generalisation of both 2-categories and monoidal categories. That being the case, there is a generalisation of lax functor, as defined above, to one, $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, in which $\mathcal{A}$ and $\mathcal{B}$ can be bicategories. (The formulation is **left to you** for later, when you have seen the definition of lax monoidal functor. It needs some more precision on the notion of bicategory so as to introduce notation for the 'associator' 2-cell, and the left and right unit 2-cells, and then a little thought on how to adapt 'lax monoidal functor' to 'lax functor' in that more general sense.)

(iv) The notion of *pseudo-functor* between 2-categories or, more generally, between bicategories is, as was said earlier, the special case of a lax functor in which the two types of 2-cell, both the $c_{f,g}$ and the c_A, are invertible.

(v) Of importance below will be the notion of an 'op-lax functor', $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, in which the arrow of the 2-cells is reversed, so $c_{f,g} : F(gf) \Rightarrow F(g)F(f)$, etc. This can be accommodated within the system of theory of lax functors by the simple device of forming, from a 2-category, $\mathcal{B}$ (or more generally), a new 2-category, $\mathcal{B}^{(2op)}$, with each $\mathcal{B}^{(2op)}(A, B) = \mathcal{B}(A, B)^{op}$, so reversing the direction of the 2-cells (and hence the notation: '(2op)' = 'opposite on 2-cells'). With this, an op-lax functor, $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, is just a lax functor $\mathcal{F}^{(2op)} : \mathcal{A}^{(2op)} \to \mathcal{B}^{(2op)}$. Of course, if $\mathcal{A}$ is locally discrete, and, thus, has no non-identity 2-cells, then ..., enough said (provided that $\mathcal{F}$ is normal)! Similarly, if $\mathcal{F}$ is a pseudo-functor, then it is both lax and op-lax, or, more precisely, it determines both a lax and an op-lax functor.

Examples: We have already seen some examples of lax, op-lax or pseudo functors, so will not give more here, except, of course the following. We cannot resist it.

Any crossed module gives rise to a 2-category, in fact a 2-group(oid), so it is natural, in the context of our discussion, to look at pseudo-functors between these 2-categories. (Why not 'lax' or 'op-lax', ..., simply that all 2-cells in these 2-categories will be invertible, so the other notions all essentially reduce to 'pseudo', with adjustment being made for the order of composition, etc.) We will examine in some detail what the resulting 'weak morphisms' of crossed modules look like a bit later, but would suggest that **examination of the idea now** and **by you** would at the same time prepare the way for that later discussion *and* give you some experience of handling these ideas if you have not met them in detail before.

Given all this about lax/op-lax and pseudo-functors, how does this relate to homotopy coherence? To examine this, let us look at homotopy coherent diagrams in a 2-category. We noted earlier (page 146) that any 2-category, $\mathcal{C}$, could be considered as an $\mathcal{S}$-category, $\mathcal{C}_\Delta$. (We should note in passing that, as each $\mathcal{C}(A,B)$ is a category, $\mathcal{C}_\Delta(A,B)$, which is just the nerve of $\mathcal{C}(A,B)$, will *not* usually be a Kan complex, but will always be a weak Kan complex / quasi-category.)

Suppose $\mathbb{A}$ is a category and $\mathcal{B}$ a 2-category (which we will consider as an $\mathcal{S}$-category, $\mathcal{B}_\Delta$, in the above way, but will not write the suffix most of the time). Let $F : S(\mathbb{A}) \to \mathcal{B}_\Delta$ be a $\mathcal{S}$-functor, and thus a homotopy coherent diagram of type $\mathbb{A}$ in $\mathcal{B}$. We have F gives:

- to each object A of $\mathbb{A}$, an object $F(A)$ of $\mathcal{B}$;
- to each pair of objects, A_0, A_1, and each $f : A_) \to A_1$, a morphism / 1-cell, $F(f) : F(A_0) \to F(A_1)$;
- to each composable pair (f,g) in $\mathbb{A}$, $\ldots$, what?

A composable pair like this corresponds to a 2-simplex

$$\begin{tikzcd} & {} \arrow[dr, "g"] & \\ {} \arrow[ur, "f"] \arrow[rr, "gf"'] & & {} \end{tikzcd}$$

in the nerve of $\mathbb{A}$, so to a functor, $\ulcorner(f,g)\urcorner : [2] \to \mathbb{A}$, which will induce $S(\ulcorner(f,g)\urcorner) : S[2] \to S(\mathbb{A})$, and, composing that with F gives

$$\begin{tikzcd} & {} \arrow[dr, "F(g)"] & \\ {} \arrow[ur, "F(f)"] \arrow[rr, "F(gf)"'] & & {} \end{tikzcd}$$

with a 2-cell, $c_{f,g} : F(gf) \Rightarrow F(g)F(f)$. This looks like it is the data for an op-lax functor. We need to check dimension 3, and a composable triple, (f,g,h), gives a diagram $[3] \to \mathbb{A}$, and hence a tetrahedral diagram in $\mathcal{B}$, when mapped by F:

$$S[3] \to S(\mathbb{A}) \to \mathcal{B}.$$

This diagram 'really lives' in the category $\mathcal{B}(F(A_0), F(A_3))$, where $A_0 \xrightarrow{f} A_1 \xrightarrow{g} A_2 \xrightarrow{h} A_3$, and is a square

$$\begin{tikzcd} F(h)F(gf) \arrow[r, Rightarrow] & F(h)F(g)F(f) \\ F(hgf) \arrow[u, Rightarrow] \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & F(hg)F(f) \arrow[u, Rightarrow] \end{tikzcd}$$

with a diagonal, and, as there are no non-trivial 3-cells in $\mathcal{B}$, there are no non-trivial 2-simplices in $\mathcal{B}(F(A_0), F(A_3))$ (either thought of as a category or as the associated simplicial set). As a result, we can conclude that the square commutes.

We thus have that a h. c. functor, $F : \mathbb{A} \to \mathcal{B}_\Delta$, reverting to the full notation, is exactly the same as a *normal* op-lax functor from $\mathbb{A}$, considered as a locally discrete 2-category, to $\mathcal{B}$.

We can note also that this gives a way of defining a *nerve for a 2-category.*

Definition: If $\mathcal{B}$ is a 2-category, we define its *nerve* to be $Ner_{h.c.}(\mathcal{B}_\Delta)$. We will write it $Ner(\mathcal{B})$.

This nerve functor has been studied by Blanco, Bullejos, and Faro, [19] and by Bullejos and Cegarra, [35] and is a specialisation of Duskin's nerve of a bi-category, [57]. Other work on this includes Gurski, [68], who links the construction with Verity's complicial sets, which we mentioned earlier. Here we will explore its properties and applications a bit more. This nerve, and also that extension of it to bicategories, is sometimes called the *Duskin nerve* of the 2-category or sometimes its *geometric nerve.*

Of course, if $\mathcal{B}$ is locally discrete, i.e., is a category masquerading as a 2-category, then $Ner(\mathcal{B})$ is just the nerve of that category.

In general, the vertices of $Ner(\mathcal{B})$ are the objects of $\mathcal{B}$, whilst the 1-simplices are the morphisms. The two simplices are diagrams of the form

$$\begin{tikzcd} & \cdot \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ \cdot \arrow[ur] \arrow[rr] & {} & \cdot \end{tikzcd}$$

and the 3-simplices correspond to tetrahedra with one of these 2-simplices in each face, hence together satisfying a cocycle condition. Above that dimension, as we will see, things are determined by their 3-skeletons.

Remarks: We could derive at least two other nerves from this construction, both of which give useful information on $\mathcal{B}$.

(i) We could define a nerve using lax rather than op-lax functors from the various $[n]$ to $\mathcal{B}$. In this case, the basic 2-simplex would look like

$$\begin{tikzcd} & \cdot \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ \cdot \arrow[ur] \arrow[rr] & {} & \cdot \end{tikzcd}$$

This variant does need mentioning, but its detailed treatment will not differ greatly from that of the geometric nerve, since it is $Ner(\mathcal{B}^{(2op)})$. If we need it, we can write it in that form or introduce as a shorthand, $Ner_{lax}(\mathcal{B})$.

(ii) We could also restrict attention to a 'pseudo'-version of this geometric nerve, in which the 2-cell is specified to be invertible:

$$\begin{tikzcd} & \cdot \arrow[dr] \arrow[d, phantom, "\Uparrow\cong"] & \\ \cdot \arrow[ur] \arrow[rr] & {} & \cdot \end{tikzcd}$$

This is related to the 2-nerve of a bicategory as considered by Lack and Paoli, [87]. We will not need to use this explicitly as the nearest we get to it has $\mathcal{B}$ a 2-groupoid - so all its 2-cells are invertible. It is important, however, to note that passing between $Ner_{lax}(\mathcal{B})$ and $Ner_{lax}(\mathcal{B})$, one does not get an isomorphic simplicial set. This pheomenon can already be seen for nerves of groupoids. If you take, say, a 2-simplex in the nerve of a groupoid and then form the corresponding 2-simplex with the inverses you get the conjugate 2-simplex and this is not giving an automorphism of the nerve as it is incompatible with the face maps.

What sort of properties does this geometric nerve functor have? What should we intuitively expect, so some idea could guide our investigations?

For a small category $\mathbb{C}$, $Ner(\mathbb{C})$ has some very interesting and useful properties, (see the discussion around about page 155). We pick out that, if we have a k-simplex, σ in $Ner(\mathbb{C})$ with $k > 1$, then σ is completely determined by its 1-skeleton. Its 1-skeleton encodes not only that the various edges fit together, but each triangular face of σ records the fact that the d_1-face is the composite of the other two. We saw this in section 5.4.2. We can formalise this in other terms using the terminology of an earlier section, **??** (especially page **??**). For any $k > 2$, and in any diagram

$$\begin{tikzcd} \partial\Delta[k] \arrow[r] \arrow[d] & Ner(\mathbb{C}) \\ \Delta[k] \arrow[ur, dashed] & \end{tikzcd}$$

there is a unique choice of dotted arrow. Remember that this is referred to as follows:

Lemma 32 *For any small category, $\mathbb{C}$, $Ner(\mathbb{C})$ is a 2-coskeletal simplicial set.*

Proof: Suppose that we have the shell, $x = (x_0, x_1, x_2, x_3)$ of a possible 3-simplex, i.e.,

$$x : \partial\Delta[3] \to Ner(\mathbb{C}),$$

then we have the individual 'faces', x_i that fit together correctly. For instance, x_3 is the 'face missing out 3', i.e.,

$$\begin{tikzcd} & x(1) \arrow[dr, "x(12)"] & \\ x(0) \arrow[ur, "x(01)"] \arrow[rr, "x(02)"'] & & x(2) \end{tikzcd}$$

and, as this is in $Ner(\mathbb{C})$, this means $x(02) = x(12)x(01)$, and so on. We thus have

$$x(03) = x(23)x(02) = x(23)x(12)x(01).$$

The only 3-simplex that will work is, of course, $\sigma := (x(01), x(12), x(23))$ and so, in the diagram

$$\begin{tikzcd} \partial\Delta[3] \arrow[r, "x"] \arrow[d] & Ner(\mathbb{C}) \\ \Delta[3] \arrow[ur, dashed, "\sigma"'] & \end{tikzcd}$$

this σ works and is the only choice. Of course, the same is true in higher dimension replacing 3 by k. (You are **left to prove** the general form of this, e.g. by induction or directly.) ■

What about $Ner(\mathcal{C})$, when $\mathcal{C}$ is a 2-category? We might guess the following:

Proposition 39 *For any (small) 2-category, $\mathcal{C}$, $Ner(\mathcal{C})$ is a 3-coskeletal simplicial set.*

Proof: We assume given $x = (x_0, x_1, x_2, x_3, x_4)$, the shell of a potential 4-simplex, and hence

$$\begin{tikzcd} \partial\Delta[4] \arrow[r, "x"] \arrow[d] & Ner(\mathcal{C}) \\ \Delta[4] \arrow[ur, dashed, "?\sigma"'] & \end{tikzcd}$$

and try to see how to build the dotted arrow, σ, so $x_i = d_i\sigma$ for each of the indices, i. The simplest way to do this is to see what makes up such a σ. It is a h.c. diagram of type [4] in $\mathcal{C}$ corresponding, therefore, to an $\mathcal{S}$-functor,

$$\sigma : S[4] \to \mathcal{C},$$

and we discussed $S[4]$ in section 5.4.5. The key diagram is a cube in the *category*, $\mathcal{C}(x(0), x(4))$. That cube needs to commute as there are no non-identity 2-cells in $\mathcal{C}(x(0), x(4))$. We saw (again in section 5.4.5) that, of the 6 faces of this cube, 5 come from the 5 faces of the 4-simplex, hence, if σ is to complete the diagram, these 5 faces must coincide with those specified by the x_i for $i = 0, 1, \ldots, 4$. In other words, we have, within x, the information on all but one face of that cube. Each of those faces is commutative as it comes from a $x_i : S[3] \to \mathcal{C}$. What about the 'extra face'? This is (using the same sort of notation as before):

$$\begin{tikzcd} x(34)x(23)x(02) \arrow[r, Rightarrow, "x(34)x(23)x(012)"] & x(34)x(23)x(12)x(01) \\ x(24)x(02) \arrow[u, Rightarrow, "x(234)x(02)"] \arrow[r, Rightarrow, "x(24)x(012)"'] & x(24)x(12)x(01) \arrow[u, Rightarrow, "x(234)x(12)x(01)"'] \end{tikzcd}$$

but the commutativity of such a diagram, in general, is equivalent to the interchange law holding in $\mathcal{C}$:

$$\begin{tikzcd} \cdot \arrow[r, bend left, "x(02)"] \arrow[r, bend right, "x(12)x(01)"'] \arrow[r, phantom, "\Downarrow x(012)"] & \cdot \arrow[r, bend left, "x(24)"] \arrow[r, bend right, "x(34)x(23)"'] \arrow[r, phantom, "\Downarrow x(234)"] & \cdot \end{tikzcd},$$

which, of course, it does.

It follows that, given x, we already have all the information needed to specify a unique σ, which completes the proof. ∎

The following could have been mentioned much earlier, but was not needed until now:

Proposition 40 *The nerve functor,*

$$Ner : Cat \to \mathcal{S},$$

is full and faithful.

Proof: The 'reason' for this result is that all the information on a (small) category, $\mathbb{C}$, is contained in the first few levels of its nerve, $Ner(\mathbb{C})$. The objects are the vertices and thus form $Ner(\mathbb{C})_0$; the 1-simplices are simply the arrows, so levels 0 and 1 give, together with the face maps and degeneracies, the basic *combinatorial* structure of $\mathbb{C}$. For the composition, one uses $Ner(\mathbb{C})_2$, of course, and the fact the $Ner(\mathbb{C})$ is 2-coskeletal.

That is the 'reason', now for the proof!

We have to examine the function

$$Ner(\mathbb{C})_{\mathbb{C},\mathbb{D}} : Cat(\mathbb{C},\mathbb{D}) \to \mathcal{S}(Ner(\mathbb{C}), Ner(\mathbb{D})),$$

for $\mathbb{C}$, $\mathbb{D}$ arbitrary small categories. (Check back for 'full' and 'faithful' on page **??** if you have forgotten their meanings.)

This is largely a question of routine checking. If $f : Ner(\mathbb{C}) \to Ner(\mathbb{D})$ is a simplicial map, then f_0 is an assignment

$$f_0 : Ob(\mathbb{C}) \to Ob(\mathbb{D})$$

and

$$f_1 : Arr(\mathbb{C}) \to Arr(\mathbb{D})$$

compatibly with source and target maps, so f has the combinatorial structure necessary for a functor. Compatibility with composition is a consequence of f_2 and *its* compatibility with the face maps. Preservation of identities is obvious, and f defines a functor from

$$F : \mathbb{C} \to \mathbb{D}$$

from which, on applying Ner, we get back f itself. We thus have that $Ner(\mathbb{C})_{\mathbb{C},\mathbb{D}}$ is surjective. In fact, better than that, we have constructed an inverse for it, so it is bijective. (Of course, there are some minor **checks to do**, but these are straight forward.) ■

This says that, in many ways, Cat behaves like a subcategory of $\mathcal{S}$ and this is one of the intuitions that fit well with our categorification process. It motivates quasi-categories and complicial sets, both models for certain classes of weak infinity categories and weak infinity categories are one way of trying to understand cohomology in the general non-Abelian setting.

What about 2-categories? Is the nerve from $2-Cat$ to $\mathcal{S}$ full and faithful? In some ways, we should *not* expect it to be. It is defined using lax / homotopy coherent functors, so we should expect it to reflect that somewhere. There is also a less explicit reason for suspecting that it would not be full and faithful. It 'feels' as if $2-Cat$ is not a complete 'categorification' of Cat. Categorification' certainly involves replacing sets by categories, functions by functors, etc., as in the passage from Cat to $2-Cat$, but also involves weakening 'equality' to 'equivalence'. Composition and identities should become weakened, so bicategories form a fuller categorification of Cat than do 2-categories. Duskin, [57], has given a generalisation of the nerve to bicategories, and this has been pushed further by Lack and Paoli, [87]. We will not go that far. (Further material can be found in the articles [18, 19, 35].)

This suggests, perhaps, that we look at Ner from the point of view of lax / op-lax / pseudo functors.

First recall that a *normal* op-lax functor, $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an op-lax functor that preserves the identities.

Lemma 33 *A normal op-lax functor, $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, between 2-categories, induces a simplicial mapping, $Ner(\mathcal{F}) : Ner(\mathcal{A}) \to Ner(\mathcal{B})$.*

Proof: We will give this 'as is', i.e., without that much reflection on what makes it work. That we will return to afterwards.

We write $\mathcal{F} = (F, c)$, as above, where F is the assignment on objects, and also denotes, sometimes with suffices, as in F_{A_0,A_1}, the functor between the relevant hom-categories, whilst c assigns 2-cells to composable pairs.

As a lax functor is neatly defined on objects and arrows, there is no problem in defining $Ner(\mathcal{F})_i$ for $i = 0$ and 1. Moreover, as $Ner(\mathcal{A})$ and $Ner(\mathcal{B})$ are 3-coskeletal, if we can define $Ner(\mathcal{F})$ in dimension 2, then it can be automatically generated in higher dimensions, since, for $k \geq 3$, any k-simplex in $Ner(\mathcal{B})$ is determined by its 2-skeleton. We thus have to concentrate on dimension 2.

A 2-simplex, σ, in $Ner(\mathcal{A})$ consists of a 4-tuple $\sigma = (\sigma(12), \sigma(02), \sigma(01); \sigma(012))$, that is, of three arrows in $\mathcal{C}$ fitting together in a triangle, together with a 2-cell filling that triangle:

$$\begin{tikzcd}
& A_1 \arrow[dr, "\sigma(12)"] \arrow[d, phantom, "\Uparrow"] & \\
A_0 \arrow[ur, "\sigma(01)"] \arrow[rr, "\sigma(02)"'] & \, & A_2
\end{tikzcd}$$

with $\sigma(012) : \sigma(02) \Rightarrow \sigma(12)\sigma(01)$ in $\mathcal{A}(A_0, A_2)$. The op-lax functor F assigns to the composable pair, $(\sigma(01), \sigma(12))$, a 2-cell

$$c_{\sigma(01),\sigma(12)} : F(\sigma(01)\sigma(12)) \Rightarrow F(\sigma(01))F(\sigma(12)),$$

and also a functor,

$$F_{02} : \mathcal{A}(A_0, A_2) \to \mathcal{B}(F(A_0), F(A_2)),$$

which, consequently, gives

$$F(\sigma(012)) : F(\sigma(02)) \Rightarrow F(\sigma(12)\sigma(01))$$

These fit together as follows:

$$\begin{tikzcd}
& F(A_1) \arrow[dr, "F(\sigma(12))"] \arrow[d, phantom, "\Uparrow"] & \\
F(A_0) \arrow[ur, "F(\sigma(01))"] \arrow[rr, bend left=15, ""{name=U}] \arrow[rr, bend right=15, "F(\sigma(02))"', ""{name=D}] & \, & F(A_2)
\arrow[from=D, to=U, phantom, "\Uparrow"]
\end{tikzcd}$$

We look at the composite 2-cell and, of course, it forms, with the other data, a 2-simplex that we take as $Ner(\mathcal{F})(\sigma)$. More formally

$$Ner(\mathcal{F})(\sigma) = (F(\sigma(12)), F(\sigma(02)), F(\sigma(01)); \alpha),$$

where $\alpha = c_{\sigma(01),\sigma(12)} \sharp_1 F(\sigma(012))$.

It is clear that this satisfies the requirements for the face maps of the nerves and the degeneracy maps work as well, since $\mathcal{F}$ is assumed to be a *normal* op-lax functor. ■

Because of this, it is clear that, considered as a functor defined on the category, $2-Cat$, of 2-categories and (strict) 2-functors, Ner cannot be full, but suppose we define a new category $2-Cat_{op-lax}$ with the same objects, *but with the normal op-lax functors* as the morphisms between them. The above lemma shows that Ner extends to a functor, $Ner : 2-Cat_{op-lax} \to \mathcal{S}$. Is this full and faithful?

Let us examine a simplicial map $f : Ner(\mathcal{A}) \to Ner(\mathcal{B})$. Can we construct an op-lax functor from it? We certainly have an assignment, F, on objects and on 1-cells, given by f_0 and f_1 respectively. For any pair, $x(01) : A_0 \to A_1$, $x(12) : A_1 \to A_2$, we have a composite $x(02) := x(12)x(01)$ and the identity 2-cell, $id : x(02) \Rightarrow x(12)x(01)$, written in that way for convenience. This gives a 2-simplex, $(x(12), x(02), x(01); id) \in Ner(\mathcal{A})_2$ and hence a 2-simplex, $f_2(x(12), x(02), x(01); id) \in Ner(\mathcal{B})_2$. We know, since f_2 is compatible with face maps, that this 2-simplex has the form $(f_1x(12), f_1x(02), f_1x(01); y) \in Ner(\mathcal{A})_2$, where y is some 2-cell,

$$y : f_1x(02) \Rightarrow f_1x(12)f_1x(01),$$

and so it is sensible to take $\mathcal{F} = (F, c)$, as suggested above, where, abusing notation slightly, $F(A) = f_0(A)$,

$$F_{A_0,A_1} : \mathcal{A}(A_0, A_1) \to \mathcal{B}(FA_0, FA_1)$$

is defined on objects by f_1, i.e., $F(x) = f_1(A_0 \xrightarrow{x} A_1)$, (but we still need F on 2-cells or, if you prefer, on the arrows in the $\mathcal{A}(A_0, A_1)$), and

$$c_{x(01),x(12)} = y,$$

as in the 2-simplex above.

We are, thus, left to define the F_{A_0,A_1} on the 2-cells and to check that they give a functor, etc.

Suppose

$$\begin{tikzcd} A_0 \arrow[r, bend left, "t(\alpha)"] \arrow[r, bend right, "s(\alpha)"'] \arrow[r, phantom, "\Uparrow\alpha"] & A_1 \end{tikzcd} ,$$

is a 2-cell of $\mathcal{A}$, then

$$\begin{tikzcd} & A_1 \arrow[dr, equal] \arrow[d, phantom, "\Uparrow\alpha"] & \\ A_0 \arrow[ur, "t(\alpha)"] \arrow[rr, "s(\alpha)"'] & {} & A_1 \end{tikzcd}$$

is a 2-simplex, $\sigma = (id, s(\alpha), t(\alpha); \alpha)$, of $Ner(\mathcal{A})$ and we get $f_2(\sigma) = (id, f_1s(\alpha), f_1t(\alpha); F(\alpha))$, defining $F(\alpha)$. (Note we are using that f is compatible with degeneracies here, and can deduce the resulting op-lax functor is going to be a normal one, i.e., identity preserving.)

We have to check that, thus defined, $F_{A_0,A_1} : \mathcal{A}(A_0, A_1) \to \mathcal{B}(FA_0, FA_1)$ *is* a functor. We suppose that we have composable two cells

$$\begin{tikzcd} A_0 \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Uparrow\beta"] \arrow[r, phantom, shift right=3, "\Uparrow\alpha"] & A_1 \end{tikzcd} ,$$

and have to compare $F(\beta\alpha)$ with $F(\beta)F(\alpha)$. To do this, we construct a 3-simplex in $Ner(\mathcal{A})$ that we will call τ, with faces:

$$\begin{aligned}
d_0\tau &= (id_{A_1}, id_{A_1}, id_{A_1}; id)\\
d_1\tau &= (id_{A_1}, s(\alpha), t(\alpha); \alpha)\\
d_2\tau &= (id_{A_1}, s(\alpha), t(\beta); \beta\alpha)\\
d_3\tau &= (id_{A_1}, s(\beta), t(\beta); \beta)
\end{aligned}$$

which, thus, fit together, diagrammatically, as:

odd numbered faces

$$\begin{tikzcd}
1 \arrow[r, "="] \arrow[dr, phantom, "\Nwarrow\beta", near start] & 2 \arrow[d, "="] \\
0 \arrow[u, "t(\beta)"] \arrow[r, "s(\alpha)"'] \arrow[ur] \arrow[ur, phantom, "\Uparrow\alpha"', near end] & 3
\end{tikzcd}$$

even numbered faces:

$$\begin{tikzcd}
1 \arrow[r, "="] \arrow[dr, "="] & 2 \arrow[d, "="] \\
0 \arrow[u, "t(\beta)"] \arrow[r, "s(\alpha)"'] \arrow[ur, phantom, "\beta\alpha\Uparrow", near start] & 3
\end{tikzcd}$$

As $Ner(\mathcal{A})$ is 3-coskeletal, (or, alternatively, because $\mathcal{A}$ has no non-trivial 3-cells!), this determines a 3-simplex, τ, as promised. Now we map this across to $Ner(\mathcal{B})$ and we get

$$F(\beta\alpha) = F(\beta)F(\alpha),$$

as expected. In other words, F_{A_0,A_1} is a functor.

The obvious question to ask now is whether or not $Ner(\mathcal{F})$ gives us back f. The way $\mathcal{F}$ was constructed on objects and at the object level of each F_{A_0,A_1} gives back f_0 and f_1 fairly obviously, so the crucial examination will be in dimension 2, '3-coskeletal-ness' handling higher dimensions.

Suppose $\sigma = (\sigma(12), \sigma(02), \sigma(01); \alpha)$ is in $Ner(\mathcal{A})$. Consider the 3-simplex, that we will denote by τ, having faces

$$\begin{aligned}
d_0\tau &= (\sigma(12), \sigma(12)\sigma(01), \sigma(01); id)\\
d_1\tau &= (id, \sigma(02), \sigma(12)\sigma(01); \alpha)\\
d_2\tau &= s_1d_0\sigma = (\sigma(12), \sigma(12), id; id)\\
d_3\tau &= (\sigma(12), \sigma(02), \sigma(01); \alpha) = \sigma,
\end{aligned}$$

(do **check that this is a 3-simplex of** $Ner(\mathcal{A})$). Map it over to $Ner(\mathcal{B})$ using f. The resulting

$f(\tau)$ has

$$\begin{aligned}
d_0 f\tau &= (f_1(\sigma(12)), f_1(\sigma(12)\sigma(01)), f_1(\sigma(01)); c)\\
d_1 f\tau &= (id, f_1(\sigma(02)), f_1(\sigma(12)\sigma(01)); F(\alpha))\\
d_2 f\tau &= s_1 d_0 f(\sigma) = (f_1(\sigma(12)), f_1(\sigma(12)), id; id)\\
d_3 f\tau &= (f_1(\sigma(12)), f_1(\sigma(02)), f_1(\sigma(01)); F(\alpha))\alpha) = f_2(\sigma).
\end{aligned}$$

Here the first use of $F(\alpha)$, as the 2-cell of $d_1 f(\tau)$, is 'by definition', whilst its occurrence as the 2-cell of $d_3 f\tau$ is deduction from the fact that $f(\tau)$ is a 3-simplex of $Ner(\mathcal{B})$. We have proved (bar invoking the 3-skeletal nature of the nerves, so as to complete the final check) that

Proposition 41 *Given any simplicial map $f : Ner(\mathcal{A}) \to Ner(\mathcal{B})$, there is a normal op-lax functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ for which $Ner(\mathcal{F}) = f$.* ■

In fact, as the data for $\mathcal{F}$ is uniquely determined by that for f, and conversely, we have the more detailed statement:

Proposition 42 *The nerve construction gives a full and faithful functor*

$$Ner : 2{-}Cat_{op-lax} \to \mathcal{S}.$$

■

This only addresses the basic level of information. In $\mathcal{S}$, we have a lot of extra 'layers' of structure, homotopies, homotopies between homotopies, etc., as $\mathcal{S}$ is an $\mathcal{S}$-enriched category. The category $2-Cat$ is also $\mathcal{S}$-enriched, as we have been using for some pages now, so what about $2{-}Cat_{op-lax}$? Are there analogues of natural transformations here, as there certainly are in $2{-}Cat$ itself? What are those analogues in this op-lax context? Do they behave nicely with respect to this nerve construction? (Recall that with Cat, natural transformations correspond to homotopies under Ner, so that seems a good question to ask in this wider context.) We need a definition of a (normal) lax transformation suitable for this setting. (We adapt this from Blanco, Bullejos and Faro, [18], as their treatment is explicitly linked to cohomological applications.)

Definition: Given two normal op-lax functors, $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{A} \to \mathcal{B}$, with $\mathcal{F}_i = (F_i, c_i)$ for $i = 1, 2$, a *(op)-lax transformation*, or *(op)-lax natural transformation*, from $\mathcal{F}_1$ to $\mathcal{F}_2$ is a pair, $\alpha = (\alpha, \tau)$, where

(i) α assigns to each object A of $\mathcal{A}$, an arrow

$$\alpha_A : F_1 A \to F_2 A$$

in $\mathcal{B}$;
and

(ii) τ assigns to each pair of objects, (A_0, A_1) of $\mathcal{A}$, a natural transformation between functors from $\mathcal{A}(A_0, A_1)$ to $\mathcal{B}(F_1 A_0, F_2 A_1)$, whose value at a 1-cell, $f : A_0 \to A_1$, (which is, thus, an object of the category $\mathcal{A}(A_0, A_1)$), is a 2-cell, τ_f, in $\mathcal{B}$,

$$\tau_f : \alpha_{A_1} F_1(f) \Rightarrow F_2(f)\alpha_{A_0},$$

(so the diagram

$$\begin{tikzcd}
F_1(A_0) \arrow[r, "\alpha_{A_0}"] \arrow[d, "F_1(f)"'] \arrow[dr, phantom, "\tau_f \Nearrow"] & F_2(A_0) \arrow[d, "F_2(f)"] \\
F_1(A_1) \arrow[r, "\alpha_{A_1}"'] & F_2(A_1)
\end{tikzcd}$$

is filled by τ_f) such that, if $\eta : f \Rightarrow g$ is an arrow in $\mathcal{A}(A_0, A_1)$,

$$(F_2(\eta)\sharp_0\alpha_{A_0})\sharp_1\tau_f = \tau_g\sharp_1(\alpha_{A_1}\sharp_0 F_1(\eta)),$$

(corresponding to a diagram of the form

$$\begin{tikzcd}
{} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "F_1(\eta)\Uparrow"] \arrow[d, "\alpha_{A_0}"'] & {} \arrow[d, "\alpha_{A_1}"] \\
{} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "F_2(\eta)\Uparrow"] & {}
\end{tikzcd}$$

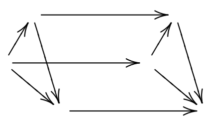
the two sides of the equation being the base and the front sides, and the top and the back, respectively).

These data are to satisfy:

1. $\tau_{1_A} = id_{\alpha_A}$ (a normalisation condition);

and

2. coherence with the structure maps, c_i, of $\mathcal{F}_i$, for $i = 1, 2$. (This is specified by a prismatic diagram: for given $A_0 \xrightarrow{f} A_1 \xrightarrow{g} A_2$, we get something like



with $c_{1;f,g}$ and $c_{2;f,g}$ on th left and right ends respectively and τ_f, τ_g and τ_{gf} on the three rectangular faces. **You are left to label the diagram yourself and thus to represent this equationally** if you wish, or need, to.)

It is often convenient, since 'op-lax natural transformation' is a bit of a mouthful, to called such a thing simply a *deformation*, (see the use in [35], for instance).

These lax natural transformations compose in a fairly obvious way, using a simple composition on the α_A-parts, and a composition of the τ_f-parts obtained by juxtaposing the resulting squares and 2-cells. This leads to a category, $OpLax(\mathcal{A}, \mathcal{B})$, of normal op-lax functors from $\mathcal{A}$ to $\mathcal{B}$, and normal lax transformations between them. This leads to:

Proposition 43 *From the category $2-Cat_{op-lax}$, and on further enriching with lax transformations, we get a 2-category.* ■

The details should be more or less clear to you, so are **left to you to complete.**

Remarks about 'pseudo' and the direction of τ: (i) There is a choice that is made when defining lax natural transformation above. The natural transformation τ_f 'measues' the extent to which the naturality square, determined by the αs, $F_1(f)$ and $F_2(f)$, does not commute, but why did it go *from* $\alpha_{A_1}F_1(f)$ *to* $F_2(f)\alpha_{A_2}$, and not the other way around. The direction is a 'convention'. It is the 'default choice' and why that choice was made is probably 'lost in time'! The opposite choice works just as well, but often in the sort of examples we consider, the choice is almost completely immaterial as the τ_f are all invertible. This happens when $\mathcal{B}$ is a 2-groupoid, rather than just a 2-category, and we will see examples in which that is the case shortly.

(ii) If one takes the definition and strengthens it by *requiring* that each τ_f be invertible, then we get a version of the definition of a *normalised pseudo-natural transformation*. The case of this where $\mathcal{A}$ is locally discrete (i.e., is just a category) is considered in Borceux and Janelidze, [22]. Of course, if $\mathcal{B}$ is a 2-groupoid, every deformation will be a pseudo-natural transformation, however it is still *important to have a direction* on the 2-cells, even though they are all invertible.

As we have said earlier, functors, which have a natural transformation between them, induce homotopic simplicial maps under the nerve functor. The natural transformation data gives the data for the homotopy. We want to see if anything similar happens with op-lax functors and deformations.

By way of a 'warm-up', we will first look at the 1-categorical case. Suppose $\alpha : F_0 \Rightarrow F_1 : \mathbb{A} \to \mathbb{B}$ is a natural transformation between functors from $\mathbb{A}$ to $\mathbb{B}$, then we have simplicial maps, $f_i = Ner(F_i) : Ner(\mathbb{A}) \to Ner(\mathbb{B})$, and want to construct a homotopy,

$$h : Ner(\mathbb{A}) \times \Delta[1] \to Ner(\mathbb{B}) \quad h : f_0 \simeq f_1,$$

(using α). Of course, α gives us a family $\{\alpha_A\}$ of 1-simplices of $Ner(\mathbb{B})$, so we can use that to define the map, h, that we want on $\langle a_1\rangle \times \Delta[1]$, for a 1-simplex $(a_1 : A_0 \to A_1)$ of $Ner(\mathbb{A})$, by the diagram:

$$\begin{tikzcd} \cdot \arrow[r, "f_1(a_1)"] & \cdot \\ \cdot \arrow[u, "\alpha_{A_0}"] \arrow[r, "f_0(a_1)"'] \arrow[ur] & \cdot \arrow[u, "\alpha_{A_1}"'] \end{tikzcd}$$

which commutes (since α is natural), so causes no difficulty on defining the diagonal. For an n-simplex, $\sigma = (A_0 \overset{a_1}{\to} A_1 \to \ldots \overset{a_n}{\to} A_n)$ in $Ner(\mathbb{A})_n$, we just repeat that recipe on each edge, getting a commutative prism, and defining h on $\sigma \times \Delta[1]$. Clearly this works, although we have left out the detailed formulae.

Now replace $\mathbb{A}$ and $\mathbb{B}$ by two 2-categories, F_0 and F_1 by op-lax functors, and α by an op-lax natural transformation. Much of the construction looks as if it works, with some modification. If we write $\alpha = (\alpha, \tau) : \mathcal{F}_0 = (F_0, c_0) \Rightarrow \mathcal{F}_1 = (F_1, c_1) : \mathcal{A} \to \mathcal{B}$, and then put $f_i = Ner(\mathcal{F}_i)$, we can adapt the diagram for h on $\langle a_1 \rangle \times \Delta[1]$ (with the same notation as above) to be

$$
\begin{tikzcd}
{} \arrow[r, "f_1(a_1)"] \arrow[dr, phantom, "\Nwarrow \tau_{a_1}" near start] & {} \\
{} \arrow[u, "\alpha_{A_0}"] \arrow[r, "f_0(\alpha_1)"'] \arrow[ur] \arrow[ur, phantom, "=" near end] & {} \arrow[u, "\alpha_{A_1}"']
\end{tikzcd}
$$

With that basic change, it is reasonably routine (i.e., a bit of intuition, plus a lot of checking!) to construct h as a homotopy defined on the 1-skeleton of $Ner(\mathcal{A})$. Given the coskeletal propertes of $Ner(\mathcal{B})$, we have to work out how to give h on $Ner(\mathcal{A})_2$, i.e., on the 'cylindrical' prisms of form $(\sigma(12), \sigma(02), \sigma(01); \sigma(12)) \times \Delta[1]$. (This is **left to you**, but first glance - in fact, stare, - at the diagram for naturality with respect to 2-cells and the coherence diagram for condition 2 of the definition of op-lax natural transformation.) Once you have done the work, you will have a proof of the following:

Proposition 44 *(see Blanco, Bullejos, Faro, [19]) Let $\mathcal{F}_0, \mathcal{F}_1 : \mathcal{A} \to \mathcal{B}$ be two normal op-lax functors between 2-categories. Every deformation, $\alpha : \mathcal{F}_0 \Rightarrow \mathcal{F}_1$, induces a homotopy, $h = Ner(\alpha) : Ner(\mathcal{F}_0) \Rightarrow Ner(\mathcal{F}_1)$.* ■

Note: due to a difference in conventions, the above reference states the direction of h to be reversed.

It is clear that, as the construction of h leads to one of the two 2-cells in each of the above diagrams being an equality, and as not every simplicial homotopy between maps from $Ner(\mathcal{A})$ to $Ner(\mathcal{B})$ would have that form, not all such homotopies can be realised by deformations. However, if we are working with 'pseudo' rather than merely 'lax' situations, for instance, if $\mathcal{B}$ is a 2-groupoid, then, in any such square,

$$
\begin{tikzcd}
{} \arrow[r, "f_1(a_1)"] \arrow[dr, phantom, "\tau_1 \Nwarrow" near start] & {} \\
{} \arrow[u, "\alpha_{A_0}"] \arrow[r, "f_0(\alpha_1)"'] \arrow[ur] \arrow[ur, phantom, "\Searrow \tau_2" near end] & {} \arrow[u, "\alpha_{A_1}"']
\end{tikzcd}
$$

we have that τ_2 is an invertible 2-cell, so we can build a new square replacing τ_2 by an identity 2-cell and τ_1 by $\tau_1 \tau_2^{-1}$, and still giving a homotopy as needed. This suggests the following result (which we **leave to you to prove more formally**).

Proposition 45 *Suppose $\mathcal{F}_i : \mathcal{A} \to \mathcal{B}$, $i = 0, 1$, are two normal op-lax functors with $\mathcal{B}$ a 2-groupoid, then, if there is a homotopy $h : f_0 \simeq f_1$, where $f_i = Ner(\mathcal{F})_i$, then there is a deformation, α, from $\mathcal{F}_0$ to $\mathcal{F}_1$, and the resulting homotopy, $Ner(\alpha)$, is homotopic to the given h.* ■

5.5.3 Weak actions of groups

This example is mostly a continuation of the previous one, but, as it is one we have considered before, and is very central to our cohomological theme, it seems a good thing to start a new section for it.

Earlier, in section **??**, we looked at the way that, in an extension of groups,

$$\mathcal{E}: \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

a section of p gave a 'lax' action' of G on K. At that point in these notes, we had not a sufficient knowledge of 'lax' or 'pseudo' ideas, nor the concepts and terminology necessary for a fuller treatment. We have now!

We start by recalling (see page 10 for starters) a little of the terminology and notation and the fundamental ideas of actions in the algebraic context. We have a group, G, and so a single object groupoid, $G[1]$. If we have a functor, $\mathcal{K}$, from $G[1]$ to $Grps$, then the functor picks out a group, $K = \mathcal{K}(*)$, where $ObG[1] = \{*\}$, and a mapping

$$\mathcal{K}_{*,*} : G[1](*, *) \to Grps(K, K) = End(K),$$

where $End(K)$ is the monoid of endomorphisms of K. The domain here is, of course, just G and the image will be within the submonoid of invertible endomorphisms, i.e., within $Aut(K)$, the group of automorphisms of K, so we get one of the usual formulations of an action of G on K, namely as a homomorphism from the group G to $Aut(K)$.

Remark: If we start with G a groupoid, then it already has a set, G_0, of objects, (and we do not need to make G into a groupoid!), then a functor $\mathcal{K} : G \to Grps$ will pick out a *family* $\{K(x) \mid x \in G_0\}$ of groups, and, if $G(x, y)$ is non-empty, morphisms between $K(x)$ and $K(y)$. (Remember G is not necessarily a connected groupoid.) Our discussion for groups extends without problem to groupoids. (A good reference for this is Blanco, Bullejos and Faro, [19], and that has been used as one source for the treatment here.)

We have seen, page **??**, that natural transformations between such functors correspond to conjugation by elements of K.

Given our interest in lax and pseudo functors and natural transformations, it is natural to look at such things in this 'action' context and to see if they correspond to anything 'well known'.

We will do this somewhat pedantically, also repeating ideas that were met earlier. We treat G, firstly, as the groupoid, $G[1]$, as before, and then as a (2-)discrete 2-category, which will also be written $G[1]$. We look at $Grps$ as a subcategory of $Grpds$ and then enrich $Grpds$ using the functor category construction, so

$$Grpds(G, H) = H^G = Func(G, H),$$

so making $Grpds$ into a 2-category, denoted Grpds. We also will need it as an $\mathcal{S}$-category via the nerves, $Ner(H^G)$.

All 2-cells in Grpds are invertible, so 'lax', 'op-lax' and 'pseudo' more or less coincide. Now for the 'deconstruction' of a lax functor, $\mathcal{K} = (K, \sigma)$,

$$\mathcal{K} : G[1] \to \mathsf{Grpds}.$$

This will correspond, according to the above definition to assignments:

- As $G[1]$ has just one object, we get a group (or more generally a groupoid), $K = \mathcal{K}(*)$, as with an action;

- For any two objects of $G[1]$ (well that is easy, both must be $*$!), a *functor*

$$\mathcal{K}_{*,*} : G[1](*,*) \to \mathsf{Grpds}(K,K),$$

where $G[1](*,*) = G$, but take care, here. Since the 2-category $G[1]$ is a locally discrete 2-category, G is also being thought of as a discrete category, that is a *set*; the vertical composition in the 2-category, i.e., of 2-cells, is necessarily trivial, the *horizontal* composition is the multiplication of the group. This just gives a family, $\{K(g) \mid g \in G\}$, of endomorphisms of K. For convenience, if $g \in G$, $K(g)$ is an endomorphism of K and we may write ${}^g k$ for $K(g)(k)$.

- For any three objects of $G[1]$ (no comment this time!), a natural transformation, σ, between 'functors' from $G[1](*,*) \times G[1](*,*)$ to $\mathsf{Grpds}(K,K)$, whose component on a pair (g_2, g_1) is a 2-cell

$$\sigma_{(g_2,g_1)} : K(g_2 g_1) \Rightarrow K(g_2)K(g_1).$$

Note that (g_2, g_1) is a composable pair of morphisms in $G[1]$! (As usual we will want $K(1_G)$ to be the identity endomorphism of K, i.e., for $\mathcal{K}$ to be *normal* and also for $\sigma_{(1,g)} = \sigma_{(g,1)} = 1_K$. As we saw when considering 'auto-equivalences', back in section **??**, such a set-up gives that each $K(g)$ is an automorphism of K, not just an endomorphism.)

The pair, $\mathcal{K} = (K, \sigma)$, must satisfy the coherence rule with the associative law, i.e., if $g_3, g_2, g_1 \in G$ (thus are composable maps in $G[1]$!), the diagram

$$\begin{tikzcd}
K(g_3g_2g_1) \arrow[r, Rightarrow, "\sigma_{(g_3g_2),g_1}"] \arrow[d, Rightarrow, "\sigma_{g_3,(g_2g_1)}"'] & K(g_3g_2)K(g_1) \arrow[d, Rightarrow, "\sigma_{g_3,g_2}K(g_1)"] \\
K(g_3)K(g_2g_1) \arrow[r, Rightarrow, "K(g_3)\sigma_{g_2,g_1}"'] & K(g_3)K(g_2)K(g_1)
\end{tikzcd}$$

commutes.

We could take thus apart further, ..., but will leave that for **you to check up** on, as we have done this all before in various forms and guises. Natural transformations correspond to conjugation (page **??**) in this context. Autoequivalences are automorphisms (same page) and so on. The coherence rule is a cocycle condition, of course.

This gives us the data for an op-lax functor,

$$\mathcal{K} : G[1] \to \mathsf{Grpds},$$

but, of course, only uses a tiny part of Grpds as it only involves one object, namely K. We have a sub 2-category, determined by K, that we will write $\mathsf{End}(K)$ as it is all the endofunctors of K and the natural transformations between them, with composition as the 'horizontal' operation. Within $\mathsf{End}(K)$, we have $\mathsf{Aut}(K)$ (and, yes, this *is* essentially the same notation as what we saw earlier, in our initial discussion of lax actions in section **??**, and even earlier, way back in section 2.1.1, except that here $\mathsf{Aut}(K)$ is the 2-group, whilst earlier we used the notation for the corresponding crossed module). This is the sub 2-category of $\mathsf{End}(K)$ whose 1-cells are the automorphisms of K. It is, as we just said, a 2-group.

We thus have that our op-lax functor, $\mathcal{K}$, is 'really' an op-lax functor

$$\mathcal{K} : G[1] \to \mathsf{Aut}(K),$$

and is also a pseudo-functor, as all 2-cells involved are invertible. (We have that last statement was true throughout our recent discussion, of course, as Grpds has all 2-cells invertible.)

Definition: Given groups, G and K, a *lax action* or *weak action* of G on K is an op-lax functor

$$\mathcal{K} : G[1] \to \mathsf{Aut}(K).$$

We can rewrite the above discussion to get more convenient forms of this.

Proposition 46 *(i) A weak action of G on K assigns, to each $g \in G$, an automorphism ${}^g(-) : K \to K$, and to each pair (g_1, g_2) in $G \times G$, an element $k = k(g_1, g_2)$ in K such that, for any $x \in K$,*

$$k.{}^{(g_2,g_1)}x = {}^{g_2}({}^{g_1}x).k,$$

(i.e., the inner automorphism by k is the difference between operation with g_2g_1 on the one hand, and with first g_1 and then g_2 on the other);
and satisfying : for all $x \in K$ and triples (g_3, g_2, g_1) of elements of G

a) ${}^1x = x$;

b) $k(1, 1) = 1$;

c) (cocycle condition)

$$k(g_3, g_2)k((g_3g_2), g_1) = {}^{g_3}k(g_2, g_1)k(g_3, g_2g_1).$$

Conversely any such assignment determines a weak action.

(ii) A weak action of G on K determines, and is determined by, a simplicial mapping

$$\mathsf{k} : Ner(G[1]) \to Ner(\mathsf{Aut}(K)).$$

Proof: (i) is just the result of taking apart the definition, and then interpreting the terms in more elementary language, so ... .

(ii) is just a corollary of our earlier result that Ner is full and faithful. ∎

This second part deserves some more comment. The domain of k is the classifying simplicial set of G, that which has been written BG in earlier chapters. (As an aside, we should note that often in earlier chapters, G was a sheaf of groups on some space, or, more generally, a group object in some topos. The corresponding theory of lax and pseudo-functors, lax natural transformations, etc., also applies there with minimal disruption / adaptation. Adapting it to the situation in which G and K are bundles of groups, i.e., bringing in a topology on them *is* somewhat harder, but can be done, as can a smooth 'Lie' theory of these.)

BEWARE: in our earlier discussion, composition order may have been reversed.

The codomain of k is interesting and raises a question. That nerve is of $\mathsf{Aut}(K)$, the 2-group of automorphisms of K, but that is, of course, the 2-group associated to the crossed module, also denoted $\mathsf{Aut}(K) = (K, Aut(K), \iota)$, that we have used so many times. Replacing $\mathsf{Aut}(K)$ by an arbitrary 2-group, $\mathcal{X}(\mathsf{C})$, corresponding to a crossed module, $\mathsf{C} = (C, P, \partial)$, we now have *two* different classifying space objects associated to it, the nerve of the associated 2-group in this 'lax' interpretation and our earlier one going via the nerve of the simplicial group (so the nerve of one of the structures, the internal groupoid one), followed by using $\bar{W}$, (recall this from sections **??** and **??**). We will return to a more detailed examination of this very shortly.

Another question that was left over from an earlier chapter, (page **??**), was of the details of the statement that a section, s, of the epimorphism

$$p : E \to G$$

in our extension

$$\mathcal{E} : \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

gave a lax action of G on K. (Another useful link at this point is to our discussion of fibred categories, for instance, in section **??**. The themes there interact with some of what we will be seeing here.) This is quite well known and is not that hard to provide in detail, so we will leave **you to do this**, but the above discussion should ease the formalisation process. Given a section $s : G \to E$, **you should construct** a lax action **in detail** either as an explicit op-lax functor, or as a simplicial map, perhaps by adapting earlier discussions and using the monadic resolution approach from section 5.2.3, mixed with more recent comments about the relationship between 2-categories and $\mathcal{S}$-categorical methods. The choice is yours and as usual, approaching it in at least two ways can clarify relationships between the approaches. (The reference mentioned above to Blanco, Bullejos, and Faro, [18], may once again help in this.)

This quite naturally, raises other questions - and again investigation is well worth it, and is **left to you**. If we change from the section, s, to another, we clearly should get a lax natural transformation between the weak actions and hence a homotopy between the corresponding simplicial maps. (Again **you are left** to search for, and give, explicit expressions for these and to link them all together into a description in terms of lax / pseudo functors, etc., the cohomology groupoid that they give, and of the equivalence classes of non-Abelian extensions that we looked at in section **??**.)

The important thing to note is how the different approaches interact and, in fact, intermesh, as this is very useful when generalising and extending things to higher dimensions and to further 'categorification'.

The end result of this investigation would be a version of the results on extensions of G by K, in terms of the set, $[Ner(G[1]), Ner(\mathsf{Aut}(K))]_*$, of (normalised) homotopy classes of pointed simplicial maps. An interesting idea **to follow up** is to link this all up with observations on 'extensions as bitorsors' (page **??**, but take care as the extension there uses different notation), the use of classifying spaces in classifying bitorsors and in particular nerves of $\mathsf{Aut}(K)$, then back to the first discussion of 'lax actions' in section **??**.

5.6 Two nerves for 2-groups

We suggested in the previous section that we have more or less 'by chance' now got two different ways of defining a nerve-like simplicial set for a 2-group, $\mathcal{X}(\mathsf{C})$, associated to a crossed module, C, and hence of assigning a 'nerve' to a crossed module. Discussion of this will take us right back to the basics of crossed modules and so it warrants a section by itself. This will also allow more easy reference to be made to the key ideas here.

We met, back in section **??**, the classifying 'space' construction, and revisited it in section **??**, which took a crossed module or its associated 2-group, thought of it as an internal category within the category of groups, constructed the (*internal*) nerve of that (*internal*) category *internally* within *Grps*, so getting a simplicial group, the *simplicial group nerve*, $K(\mathsf{C})$, of C. This was then processed further using $\overline{W}$, to get $\overline{W}(K(\mathsf{C}))$. This was analysed (on page **??**) in the slightly more general case when C is a reduced crossed complex. (Take care when reviewing those pages as the $\mathcal{S}$-groupoids are given for the algebraic composition convention.)

We also have the following chain of ideas. A 2-group, $\mathcal{X}(\mathsf{C})$, is a special type of 2-category and any 2-category, as we have just seen, gives an $\mathcal{S}$-category by taking the nerve of each 'hom'. Of course, then the natural thing to do, if we want a nerve, is to take the (homotopy coherent) nerve of that $\mathcal{S}$-category and, again of course, this is the *geometric* nerve of the 2-group. What does *it* look like?

Before we do investigate this more fully, let us see, briefly, why it is important to do so.

The route to a nerve via $\overline{W}$ has important links to simplicial fibre bundle theory; $\overline{W}$ has the Dwyer-Kan 'loop groupoid', (glance back at page **??** if need be), as a left adjoint and all the mechanisms of twisted Cartesian products, twisting functions, etc., that we looked at in section **??** are there for use. The homotopy coherent nerve, on the other hand, opens the way to interpretations of maps as homotopy coherent actions, to links with lax / op-lax / pseudo-category theory, and thus quite directly into the methods of low dimensional non-Abelian cohomology.

We will see that the two nerves are very similar; in fact, they are isomorphic. This suggests many lines of enquiry. Both constructions work for a general $\mathcal{S}$-category, so there are possibilities of links between their extensions to general $\mathcal{S}$-groupoids, or to strict monoidal categories, since they are one object 2-categories. These links have been, in part, investigated in papers by various authors, in particular, Bullejos and Cegarra, [35] and [36], Blanco, Bullejos and Faro, [19] and [18]. Some of these use, instead of $\overline{W}$, a combination of the nerve on the group structure to get a bisimplicial set, followed by using the diagonal of that 'binerve', a method related to what we saw in section **??**. The $\overline{W}$-construction corresponds to taking the nerve in the 'group direction' followed by using the Artin-Mazur codiagonal, ∇. We will look at this in some detail shortly (starting on page **??**). That the resulting constructions are weakly homotopically equivalent follows from the results of Cegarra and Remedios, [39], who prove several results generalising some unpublished work of Zisman.

Back to a detailed look at $Ner(\mathcal{X}(\mathsf{C}))$, we can, of course, just read its details off from our earlier look at $Ner(\mathcal{C})$ for $\mathcal{C}$, a 2-category, together with the description of $\mathcal{X}(\mathsf{C})$ as a 2-category. Because in this sort of calculation, it helps to have each facet 'face-up on the table', we will recall $\mathcal{X}(\mathsf{C})$ first, although we have met it many times. (This is mostly important because of the risk of a mix of conventions, for instance, on composition order.)

5.6.1 The 2-category, $\mathcal{X}(\mathsf{C})$

- The 2-category, $\mathcal{X}(\mathsf{C})$, has a single object denoted $*$;

- The set of 1-arrows, $\mathcal{X}(\mathsf{C})(*,*)_0$, is the group, P with $p_1\sharp_0 p_2 = p_1p_2$ as composition and we picture it as
$$* \xrightarrow{p_2} * \xrightarrow{p_1} *,$$
so will use *functional composition order*.

- the set of 2-arrows, $\mathcal{X}(\mathsf{C})(*,*)_1$, is the group $C \rtimes P$. We have that, if $(c,p) \in C \rtimes P$, its source is p and its target is $\partial c.p$. We picture it, in 2-category 'imagery', as
$$\begin{tikzcd} * \arrow[r, bend left=50, "p"{name=U}] \arrow[r, bend right=50, "\partial c.p"'{name=D}] \arrow[Rightarrow, from=U, to=D, "(c,p)"] & * \end{tikzcd},$$
and have a composition, $\sharp_1$, within the category $\mathcal{X}(\mathsf{C})(*,*)$, given by
$$(c', \partial c.p)\sharp_1(c,p) = (c'c, p).$$
The other composition $\sharp_0$, a 'horizontal' composition, is, as we know, the group multiplication of $C \rtimes P$:
$$(c_2,p_2)\sharp_0(c_1,p_1) = (c_2.{}^{p_2}c_1, p_2p_1),$$
(and the interchange law holds, being equivalent to the Peiffer identity).

5.6.2 The geometric nerve, $Ner(\mathcal{X}(\mathsf{C}))$

- The set of 0-simplices, $Ner(\mathcal{X}(\mathsf{C}))_0$, is the set of objects, so is $\{*\}$. (This nerve will, here, be a reduced simplicial set. Of course, if C was a crossed module of groupoids, then $Ner(\mathcal{X}(\mathsf{C}))_0$ would possibly have more elements.)

- The set of 1-simplices will be the set of arrows of $\mathcal{X}(\mathsf{C})$ and thus is P, *as a set*;

- The 2-simplices of $Ner(\mathcal{X}(\mathsf{C}))$ consist of 4-tuples, $\underline{x} = (x(12), x(02), x(01); x(012))$, as before, where the $x(ij) \in P$ and $x(012) : x(02) \Rightarrow x(12)x(01)$ is a 2-cell.

  The faces of $\underline{x}$ are $d_0\underline{x} = x(12)$, etc, as we saw before, so we will abbreviate $x(12)$ to $x_0 \in P$, etc. Writing $x := x(012)$, we then have x is a 2-cell, $x : x_1 \Rightarrow x_0\sharp_0 x_2$, the codomain being just $x_0.x_2$ in different notation, hence x has form (c, x_1) with $\partial c.x_1 = x_0.x_2$,
$$\begin{tikzcd} & \cdot \arrow[dr, "x_0"] \arrow[d, phantom, "\Uparrow x"] & \\ \cdot \arrow[ur, "x_2"] \arrow[rr, "x_1"'] & {} & \cdot \end{tikzcd}$$
  and hence $\partial c = x_0x_2x_1^{-1}$, which is clearly closely related to the form given, page **??**, for the $\overline{W}$-based version of the classifying space, but we must check how good that similarity is in detail and with consistent conventions).

- The 3-simplices of $Ner(\mathcal{X}(\mathsf{C}))$ consist of sets of arrows,

$$\{x(ij) \mid 0 \leq i < j \leq 3\},$$

and 2-cells,

$$x(ijk) \mid 0 \leq i < j < k \leq 3\},$$

with $x(ijk) : x(ik) \Rightarrow x(jk)x(ij)$, and satisfying a cocycle condition:

$$\begin{tikzcd}
x(13)x(01) \arrow[r, Rightarrow, "x(123)\sharp_0 x(01)"] & x(23)x(12)x(01) \\
x(03) \arrow[u, Rightarrow, "x(013)"] \arrow[r, Rightarrow, "x(023)"'] & x(23)x(02) \arrow[u, Rightarrow, "x(23)\sharp_0 x(012)"']
\end{tikzcd}$$

commutes.

We again rethink this in terms of C and P, using the fact that $d_0\underline{x} = (x(23), x(13), x(12); x(123) : x(13) \Rightarrow x(23)x(12))$, and so on. The i^{th} face is the term that omits i, as usual in these situations.

It is important to note at this point that between them the four faces contain *all* the $x(ij)$ and $x(ijk)$, so completely determine $\underline{x}$ itself. This is, of course, related to the condition that $Ner(\mathcal{X}(\mathsf{C}))$ is 3-coskeletal, but that condition just gives the similar result in higher dimension. (Check back on the properties of that notion as given by Proposition **??**.) This observation says that there is a unique 3-simplex with these faces, not that if you start with four 2-simplices seemingly of the right form then there will automatically exist a 3-simplex with those 2-simplices as its faces, because the 3-cocycle condition intervenes.

Write the four 2-cells as c_0, c_1, c_2, and c_3, corresponding to $d_0\underline{x}$, etc., respectively, so that

  - face (123): $\partial c_0 = x(23)x(12)x(13)^{-1}$;
  - face (023): $\partial c_1 = x(23)x(02)x(03)^{-1}$;
  - face (013): $\partial c_2 = x(13)x(01)x(03)^{-1}$;
  - face (012): $\partial c_3 = x(12)x(01)x(02)^{-1}$.

To analyse the commutativity of the square above will require us to look first at the two 'whiskered' terms:

$$x(123)\sharp_0 x(01) = (c_0, x(13))\sharp_0(1, x(01)) = (c_0, x(13)x(01)),$$

whilst

$$x(23)\sharp_0 x(012) = (1, x(23))\sharp_0(c_3, x(02)) = ({}^{x(23)}c_3, x(23)x(02)).$$

The $\sharp_1$-compositions of 2-cells correspond to multiplication in C, so the two routes around the square give

$$\begin{aligned}
(x(123)\sharp_0 x(01))\sharp_1 x(013) &= (c_0, x(13)x(01))_1(c_2, x(03)) \\
&= (c_0c_2, x(03))
\end{aligned}$$

and

$$\begin{aligned}(x(23)\sharp_0 x(012))\sharp_1 x(023) &= ({}^{x(23)}c_3, x(23)x(02))\sharp_1(c_1, x(03))\\ &= ({}^{x(23)}c_3c_1, x(03)).\end{aligned}$$

We thus have a cocycle condition:

$$c_0c_2 = {}^{x(23)}c_3c_1.$$

- Above dimension 3, everything is determined by dimension 3, as we saw that $Ner(\mathcal{X}(\mathsf{C}))$ is 3-coskeletal.

We next turn towards the construction going via the 'internal nerve' or 'simplicial group nerve'. By this route, we first construct a simplicial group, $K(\mathsf{C})$, from C. As above we will repeat that construction in great detail, so as to check consistency of conventions. The simplicial group, $K(\mathsf{C})$, is the internal nerve of the internal groupoid, $\mathcal{X}(\mathsf{C})$, and is constructed within the category of groups. (The relevant earlier discussions are in sections **??** and **??**.)

The simplicial group, $K(\mathsf{C})$, has:

- group of 0-simplices, $K(\mathsf{C})_0 = P$;
- group of 1-simplices, $K(\mathsf{C})_1 = C \rtimes P$, with, for (c_1, p), a 1-simplex, $d_0(c_1,p) = \partial c_1.p$, $d_1(c_1,p) = p$ and $s_0(p) = (1,p)$, for $p \in P$;
- group of 2-simplices, $K(\mathsf{C})_2 = C \rtimes (C \rtimes P)$, with, for (c_2, c_1, p), a 2-simplex

$$\begin{aligned}d_0(c_2,c_1,p) &= (c_2, \partial c_1.p),\\ d_1(c_2,c_1,p) &= (c_2.c_1, p),\\ d_2(c_2,c_1,p) &= (c_1,p),\end{aligned}$$

and degeneracies, $s_0(c_1,p) = (1,c_1,p)$, $s_1(c_1,p) = (c_1,1,p)$.

It is useful to repeat the diagram for (c_2, c_1, p):

$$\begin{tikzcd} & \partial c_1.p \arrow[dr, "{(c_2,\partial c_1.p)}"] & \\ p \arrow[ur, "{(c_1,p)}"] \arrow[rr, "{(c_2c_1,p)}"'] & & \partial(c_2c_1).p \end{tikzcd}$$

- for $n \geq 3$, $K(\mathsf{C})_n = C \rtimes K(\mathsf{C})_{n-1}$, with action via the projection to P, and, if $(\underline{c}, p) := (c_n, \ldots, c_1, p)$ is an n-simplex, the face morphisms are given by

$$\begin{aligned}d_0(\underline{c},p) &= (c_n, \ldots, c_2, \partial c_1.p),\\ d_i(\underline{c},p) &= (c_n, \ldots, c_{i+1}.c_i, \ldots, p) \quad \text{for } 0 < i < n,\\ d_n(\underline{c},p) &= (c_{n-1}, \ldots, c_1, p),\end{aligned}$$

whilst the degeneracy maps insert an identity.

5.6.3 $\overline{W}(H)$ in functional composition notation

We have been operating under the assumption that to hope to obtain fairly simple formulae in cocycles, nerves, etc., it may be a good idea to stick with consistent conventions, so using left actions, function composition order, and so on. This has sometimes worked! It does mean checking through to see that a given formula is consistent with the convention and *that can be tedious!* Does it matter? The answer is 'sometimes'. The mathematical *essence* of the argument *is* fully independent of the notation, but that means that a twisted arcane obscure formula may really represent something simple, and be equivalent to a much simpler transparent one, or it may really reflect some great twisted arcane mathematical form that is impossible to unravel further.

For the $\overline{W}$-construction, we have two or three levels of structure and the order of 'composition' being used is not always in evidence, so giving a consistent convention is quite tricky.

The classifying space of a group is given by the nerve of the corresponding groupoid or, if you prefer, the geometric realisation of that simplicial set. The $\overline{W}$-construction gives a classifying space for a simplicial group (or, more generally, any $\mathcal{S}$-groupoid or small $\mathcal{S}$-category). It is a generalisation of the nerve construction. It can also be derived from the nerve, since, applying the nerve functor to each dimension of a simplicial group gives a bisimplicial set and, as we have mentioned earlier, one can process such an object either using the diagonal functor (as we did in section **??**, page **??**) or, using the Artin-Mazur codiagonal that we will meet more formally in the near future (section **??**, page **??**).

If G is a groupoid, we can represent an n-simplex of $Ner(G)$ by a diagram

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \dots \xrightarrow{g_n} x_n,$$

where $t(g_i) = s(g_{i+1})$, and, in 'functional' order, by an n-tuple $\underline{g} = (g_n, \dots, g_1)$ with $d_0\underline{g} = (g_n, \dots, g_2)$, etc. In the $\overline{W}$-construction, we look at an $\mathcal{S}$-groupoid, H, and take 'composable' strings, $\underline{h} = (h_n, \dots, h_1)$, in a similar way, but with $h_i \in H_{i-1}$.

In case you think that we need $h_i \in H_i$, it is worth pausing to discuss the indexing. In a group, G, thought of as the groupoid, $G[1]$, the nerve is a reduced simplicial set, i.e., $Ner(G)_0$ has just one element, and $Ner(G)_1$ is G itself, but the arrows in $G[1]$, as a simplicially enriched groupoid, are thought of as being in dimension 0, so the dimension drops by 1. This sort of conflict of 'rival' indexation ideas is quite usual, quite confusing and quite irritating, but it is also quite easy to accept and to work with. Remember that $\overline{W}$ behaves as if it were a 'suspension' operation, whilst its left adjoint, G, behaves like a 'loops on -' construction, so we should expect shifts in 'geometric' dimension.

The problem is 'what should the face convention be?' If we look at d_0 and define it just to delete the h_1 position, then we get an invalid string, as the dimensions are wrong. The n^{th} face would work alright as that would delete h_n and the resulting string would still be valid. To get around the d_0 problem, we will adopt a definition that (i) is simple, (ii) works and, in fact, (iii) has a neat interpretation, when applied to objects such as $K(\mathsf{C})$. In addition, it seems to be the codiagonal of the bisimplicial nerve construction, but we cannot look at that aspect in detail at the moment, as we do not yet have enough detailed information on the codiagonal.

What is this marvellous convention, ...?

We take H to be an $\mathcal{S}$-groupoid, as usual, with object set, O, say:

- $\overline{W}(H)_0$ is the set, O, of objects of H;

- $\overline{W}(H)_1$ is the set of arrows of the groupoid, H_0;

and, in general,

- $\overline{W}(H)_n$ is the set of all 'composable' strings, $\underline{h} = (h_n, \ldots, h_1)$, with $h_i \in H_{i-1}$, and (for 'composable') $t(h_i) = s(h_{i+1})$ for $0 < i < n$.

The face maps are given by:

- $d_0(\underline{h}) = (d_0 h_n, \ldots, d_0 h_2)$;
- $d_i(\underline{h}) = (d_i h_n, \ldots, d_i h_{i+1}.h_i, \ldots, h_1)$ for $0 < i < n$;
- $d_n(\underline{h}) = (h_{n-1}, \ldots, h_1)$.

The degeneracy maps are given by inserting an identity in the appropriate place and using the degeneracies of H to push earlier elements of the string up one dimensions:

- $s_i(\underline{h}) = (s_i(h_n), \ldots, s_i(h_{i+1}), id_{x_i}, h_i, \ldots, h_1)$.

(Of course, you are **left to check** that this works and gives a simplicial set, etc.)

There are some obvious questions to ask:

- Does this given an isomorphic version of $\overline{W}(H)$? Possibly not, as it looks more like a conjugate version of the more standard form. It clearly has the same sort of properties, e.g., being a classifying space for H, classifying principal H-bundles if H is a simplicial group, etc., and has a geometric realisation that is homeomorphic to the standard form.
- Is it easy to visualise the n-simplices? Yes, at least in the case $H = K(\mathsf{C})$, and more generally for any 2-groupoid considered as a $\mathcal{S}$-groupoid. In fact, it works for a 2-category as well:

5.6.4 Visualising $\overline{W}(K(\mathsf{C}))$

First let us see what the 'bottom end' of $\overline{W}(K(\mathsf{C}))$ looks like.

- $\overline{W}(K(\mathsf{C}))_0$ is a point (as we have C is a crossed module of *groups*);
- $\overline{W}(K(\mathsf{C}))_1$ is isomorphic to the *set*, P, as a 1-simplex in $\overline{W}(K(\mathsf{C}))$ is an 'arrow', i.e., an element in $K(\mathsf{C}))_0$, which is the *group* P;
- A 2-simplex of $\overline{W}(K(\mathsf{C}))$ consists of a pair (h_2, h_1) with $h_i \in K(\mathsf{C})_{i-1}$, so $h_2 \in C \rtimes P$, $h_1 \in P$.

In 2-categorical from, this can be thought of as $\underline{h}$ being

$$\begin{tikzcd} {} \arrow[r, "h_1"] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow h_2"] & {} \end{tikzcd}$$

and then $d_0(\underline{h})$ deletes h_1, as we want, and takes $d_0(h_2)$ as data from h_2; at the other 'extreme', $d_2(\underline{h})$ just gives us

$$\xrightarrow{h_1}$$

and, in between, $d_1(\underline{h})$ takes the start of the 2-cell and composes it with h_1 to get $d_1(h_2).h_1$.

It is sometimes useful to draw this as a 'staircase' diagram:

$$\begin{tikzcd}
 & & {} \\
 & {} \arrow[r, "d_0h_2"] & {} \arrow[u, "="'] \\
{} \arrow[r, "h_1"'] & {} \arrow[u, "="] \arrow[r, "d_1h_2"'] \arrow[ur, phantom, "h_2"] & {} \arrow[u, "="']
\end{tikzcd}$$

and we will see this 'come into its own' importance later when looking at codiagonals.

• The 3-simplices, $\underline{h} = (h_3, h_2, h_1)$ with, again, $h_i \in K(\mathsf{C})_{i-1}$, have similar pictures. Remember h_3 is a composable pair of 2-cells, as on the right hand end:

$$\begin{tikzcd}
{} \arrow[r, "h_1"] & {} \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Uparrow a"] & {} \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2ex, "\Uparrow c"] \arrow[r, phantom, shift right=2ex, "\Uparrow b"] & {}
\end{tikzcd}$$

and the staircase, obtained by expanding out the 2-cells:

$$\begin{tikzcd}
 & & & {} \\
 & & {} \arrow[r] & {} \arrow[u, "="'] \\
 & {} \arrow[r] & {} \arrow[u, "="] \arrow[r] & {} \arrow[u, "="'] \\
{} \arrow[r] & {} \arrow[u, "="] \arrow[r] & {} \arrow[u, "="] \arrow[r] & {} \arrow[u, "="']
\end{tikzcd}$$

The staircase shows more clearly the face maps. The d_0 deletes the bottom row completely; d_1 removes the 1st row and 1st column of vertices and composes, where possible, to give

$$\begin{tikzcd}
 & & {} \\
 & {} \arrow[r] & {} \arrow[u, "="'] \\
{} \arrow[r, "d_1a\sharp_0h_1"'] & {} \arrow[u, "="] \arrow[r] \arrow[ur, phantom, "c\sharp_1b"] & {} \arrow[u, "="']
\end{tikzcd}$$

d_2 removes the 2nd row and column and composes:

$$\begin{tikzcd}
 & & {} \\
 & {} \arrow[r] & {} \arrow[u, "="'] \\
{} \arrow[r, "h_1"'] & {} \arrow[u, "="] \arrow[r] \arrow[ur, phantom, "b\sharp_0a"] & {} \arrow[u, "="']
\end{tikzcd}$$

and d_3 deletes the right hand column (and thus the top row as well).

If we go one step further down in the notation, i.e., back to the elements of C and P, then we have $\underline{h} \in \overline{W}(K(C))_2$ has form

$$\underline{h} = ((c_{2,1}, p_2), p_1)$$

with $h_2 = (c_{2,1}, p_2) \in C \rtimes P$, and so on. The picture of $\underline{h}$ is then

$$\begin{aligned} d_0(\underline{h}) &= \partial c_{2,1}.p_2 \\ d_1(\underline{h}) &= p_2.p_1 \\ d_2(\underline{h}) &= p_1 \end{aligned}$$

giving

$$\begin{tikzcd} & \cdot \arrow[dr, "\partial c_{2,1}.p_2"] \arrow[d, phantom, "(h_2{,}h_1)"] & \\ \cdot \arrow[ur, "p_1"] \arrow[rr, "p_2.p_1"'] & {} & \cdot \end{tikzcd}$$

If we match that picture with the earlier one (page 187), then

$$\begin{aligned} x_0 &\leftrightarrow \partial c_{2,1}.p_2 \\ x_1 &\leftrightarrow p_2.p_1 \\ x_2 &\leftrightarrow p_1 \end{aligned}$$

and, given $\underline{h}$, we get the geometric nerve 2-simplex,

$$(p_1, p_2.p_1, \partial c_{2,1}.p_2; (c_{2,1}, p_2.p_1)).$$

Working the other way around, given $(x_2, x_1, x_0; (c, x_1))$, gives a $\overline{W}$-based 2-simplex

$$((c, x_1 x_2^{-1}), x_2),$$

and the faces match up. (**Check this all works** - both ways - and do not forget the 'cocycle' conditions relating the x_is.) This looks good. On to dimension 3, ...

If we start with $\underline{h} = (h_3, h_2, h_1)$, where

$$\begin{aligned} h_1 &= p_1 \\ h_2 &= (c_{2,1}, p_2) \\ h_3 &= (c_{3,2}, c_{3,1}, p_3), \end{aligned}$$

we get

$$\begin{aligned} d_0(\underline{h}) &= ((c_{3,2}, \partial c_{3,1}.p_3), (\partial c_{2,1}.p_2)), \\ d_1(\underline{h}) &= ((c_{3,2}.c_{3,1}, p_3), p_2.p_1), \\ d_2(\underline{h}) &= ((c_{3,1}.{}^{p_3}c_{2,1}, p_3 p_2), p_1), \\ d_3(\underline{h}) &= ((c_{2,1}, p_2), p_1), \end{aligned}$$

and now note that given these four faces, we can reconstruct $\underline{h}$ completely, since $d_3(\underline{h})$ gives us h_2 and h_1, and we can use projections onto semi-direct factors of $C \rtimes (C \rtimes P)$ to retrieve h_3 with no bother. This means that there is a unique $\underline{h}$ with this shell - the same phenomenon that we saw with $Ner(\mathcal{X}(\mathsf{C}))$. The isomorphism that we found in levels 0, 1 and 2 can therefore be extended to dimension 3 . . . , and above by the fact that we have 3-coskeletal simplicial sets here. (We have not *actually* explicitly checked that $\overline{W}(K(\mathsf{C}))$ *is* 3-coskeletal, but the above calculation linked in with our earlier work (page **??**) should **enable you to prove this**.) We have

Proposition 47 *The two classifying spaces, $Ner(\mathcal{X}(\mathsf{C}))$ and $\overline{W}(K(\mathsf{C}))$, are naturally isomorphic.* ∎

This result suggests several questions, *some* of which we will look at shortly, others are **left to you**.

• If C and D are two crossed modules, can we interpret, algebraically, an op-lax morphism between the corresponding 2-groups, since we know that these correspond to simplicial morphisms between the corresponding nerves? This would give a sort of 'weak' morphism between the crossed modules.

• Can we extend the above *isomorphism* to the case where we have 2-categories rather than 2-groupoids? This would look unlikely, since we had to use inverses to check the isomorphism, but perhaps some weaker relationship is possible, cf., for instance, Bullejos and Cegarra, [35]. One important consequence of this is a way of comparing the two obvious ways of assigning a classifying space to a *strict monoidal category.* A monoidal category 'is' a one object bicategory, and a strict one thus corresponds to a one object 2-category. (We will look at monoidal categories is slightly more detail in a coming chapter.) The classical classifying space construction, used by Segal, [115, 116], corresponds to taking the nerve of the category structure and then that of the monoid structure and forming a simplicial set from the resulting bicomplex. The resulting space has a lot of beautiful properties, but we will not go into them here. The relevant papers directly on the comparison between this classical nerve and classifying space and that defined using the homotopy coherent nerve are by Bullejos and Cegarra, [35, 35]. One important point to note is that the Duskin geometric nerve construction which they use is also applicable to bicategories, so some of their results apply also to non-strict monoidal categories.

• Can we find a way of adapting the above proposition to handle some sort of 3-category or 3-groupoid? Perhaps starting with a 2-crossed module, we could form $\overline{W}$ of the corresponding simplicial group, since that is easy, but can we construct the h.c. nerve of such a simplicial group?

More generally:

• If we think of an $\mathcal{S}$-groupoid, G, as an $\mathcal{S}$-category, what is the geometric (h.c.) nerve of that $\mathcal{S}$-category?

5.7 Pseudo-functors between 2-groups

We will look in some detail at the first of these questions.

As crossed modules give rise to 2-groups (or, more generally, 2-groupoids) and these are 2-categories, it is natural to ask what the lax or op-lax functors between two such 2-groups look like. This can be considered both as a good illustrative example of (op-)lax functors and thus of homotopy coherence, and also as an important part of the theory of crossed modules that we have yet to explore. We will start with a basic observation and that is that, as 2-groupoids have invertible 1 and 2 arrows, there is no essential difference between lax and op-lax functors and they are both 'the same as' pseudo-functors. Of course, one has to choose a direction for the 2-cells and we will consider 'pseudo = op-lax + invertible', i.e., the structural 2-cells of a pseudo-functor will go from $F(ab)$ to $F(a)F(b)$. These pseudo-functors will be normal ones as usual.

To start with, our study will look at pseudo-functors between two 2-groups, $\mathcal{X}(\mathsf{C})$ and $\mathcal{X}(\mathsf{C}')$, where $\mathsf{C} = (C, P, \partial)$ and $\mathsf{C}' = (C', P', \partial')$, and by analysing them at the level of the groups and actions involved. Later we will examine them at the level of simplicial groups. (As usual the extension to $\mathcal{S}$-groupoids is reasonable easy to do, so will be **left to you**.)

(The material in this section is treated, in part, by Noohi in [106, 107] (and the correction avaialable as [108]) and with Aldrovandi, [2], for a sheafified version with applications to stacks. There is also a strong link with the Moerdijk-Svensson model category structure on 2-groups, for which see [100] as well as with the papers referred to in the previous section.)

5.7.1 Weak maps between crossed modules

Effectively a *weak map between crossed modules* is what is 'seen', at the level of crossed modules, of a pseudo-functor between the corresponding 2-groups. The abstract definition as given by Noohi, [106] is:

Definition: Let C and C' be crossed modules, as above. A *weak map*, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, is a pseudo-functor from $\mathcal{X}(\mathsf{C})$ to $\mathcal{X}(\mathsf{C}')$.

That probably does not say that much to you about what such a thing looks lie, so we are going to take the definition apart in various ways so as to get some feel for them.

We first use a direct attack. Consider a normal pseudo-functor:

$$F : \mathcal{X}(\mathsf{C}) \to \mathcal{X}(\mathsf{C}'),$$

then this consists of

- a set map, F_0, on objects (this is 'no big deal' as both $\mathcal{X}(\mathsf{C})$ and $\mathcal{X}(\mathsf{C}')$ have exactly one object);
- a set map, F_1, sending arrows to arrows, so giving a function,

$$f_0 : P \to P',$$

  which is not necessarily a homomorphism of groups. The obstruction to it being one is given by
- a set map, $\varphi : P \times P \to C' \rtimes P'$, so, if $p_2, p_1 \in P$, $\varphi(p_2, p_1)$ is a 2-cell from $f_0(p_2p_1)$ to $f_0(p_2)f_0(p_1)$;
- a functor

$$\begin{tikzcd} F_2 : C \rtimes P \arrow[r] \arrow[d, shift right=2] \arrow[d] & C' \rtimes P' \arrow[d, shift right=2] \arrow[d] \\ P \arrow[u, shift right=2] \arrow[r] & P' \arrow[u, shift right=2] \end{tikzcd}$$

  between the underlying categories, with f_0 at the level of objects. (Importantly, note that this does not mean that this functor preserves horizontal composition, i.e., group multiplication, in either the top or the object levels. This is just $F_2 = F_{*,*} : \mathcal{X}(\mathsf{C})(*,*) \to \mathcal{X}(\mathsf{C}')(*,*)$, as a functor between the corresponding 'hom-categories'.)

Of course, we will have to give some equations and conditions on these, but will explore this little-by-little before giving a résumé of the resulting structure.

First we note that, as we have a normalised pseudo-functor, $f_0(1) = 1$ and $\varphi(1,1) = 1$. As F_2 is a functor, we have, for $(c,p) \in C \rtimes P$,

$$F_2(c,p) : f_0(p) \to f_0(\partial c.p),$$

but this means that $F_2(c,p)$ has the form,

$$F_2(c,p) = (F_2'(c,p), f_0(p)),$$

for some function $F_2' : C \rtimes P \to C'$. We will set $f_1(c) = F_2'(c,1)$ and note that $\partial f_1(c) = f_0(\partial c)$.

It will eventually turn out that f_1 is *almost* a group homomorphism and that from f_1 and φ, we will be able to calculate $F_2'(c,p)$ for a general $p \in P$, that is to say, the information needed for F_2 reduces to that for f_1 and φ and, from them, we can reconstruct F_2 itself.

We also have that $\varphi(p_2,p_1)$ is a 2-cell from $f_0(p_2p_1)$ to $f_0(p_2)f_0(p_1)$. It therefore has the form

$$\varphi(p_2,p_1) = (\langle p_2,p_1\rangle, f_0(p_2p_1))$$

for some 'pairing function',

$$\langle\ \ ,\ \ \rangle_\varphi : P \times P \to C'.$$

(We will usually write $\langle\ \ ,\ \ \rangle$ instead $\langle\ \ ,\ \ \rangle_\varphi$ if no confusion is likely.) We need φ to be 'natural' with respect to pre- and post- whiskering and so will have corresponding conditions on $\langle\ \ ,\ \ \rangle$. We first note that, since the target of $\varphi(p_2,p_1)$ is $f_0(p_2)f_0(p_1)$, we have

Lemma 34 ***(Target condition)*** *For any $p_1, p_2 \in P$,*

$$\partial\langle p_2,p_1\rangle = f_0(p_2)f_0(p_1)f_0(p_2p_1)^{-1}.$$

■

The 'associativity' axiom for φ gives a cocycle condition:

for $p_1,p_2,p_3 \in P$, the diagram, in $\mathcal{X}(\mathsf{C}')$

$$\begin{tikzcd}
f_0(p_3p_2p_1) \arrow[r, "\varphi(p_3p_2{,}p_1)"] \arrow[d, "\varphi(p_3{,}p_2p_1)"'] & f_0(p_3p_2)f_0(p_1) \arrow[d, "\varphi(p_2{,}p_1)\sharp_0 f_0(p_1)"] \\
f_0(p_3)f_0(p_2p_1) \arrow[r, "f_0(p_3)\sharp_0\varphi(p_2{,}p_1)"'] & f_0(p_3)f_0(p_2)f_0(p_1)
\end{tikzcd}$$

is commutative.

Interpreting this at the crossed module level:

Lemma 35 ***(Cocycle condition)*** *For any $p_1, p_2, p_3 \in P$,*

$$\langle p_3, p_2\rangle\langle p_3p_2, p_1\rangle = {}^{f_0(p_3)}\langle p_2, p_1\rangle\langle p_3, p_2p_1\rangle.$$

■

The proof is straightforward. We note that we really do use the formulae for pre- and post-whiskering in terms of the group multiplication. This is just the multiplication on the right or left of (c, p) by some $(1, p')$:

Pre-whisker: $(c, p)\sharp_0(1, p') = (c, pp')$;
Post-whisker: $(1, p')\sharp_0(c, p) = ({}^{p'}c, p'p)$.

As we are considering normalised op-lax and pseudo- functors, we have $\varphi(1, 1) = 1$, so $\langle 1, 1\rangle = 1$ as well, but we can use this together with the cocycle condition to get:

Corollary 10 *For any $p \in P$, $\langle 1, p\rangle$ and $\langle p, 1\rangle$ are both $1_{C'}$.*

Proof: Taking $p_2 = p$, $p_3 = 1$ and $p_1 = p^{-1}$ gives

$$\langle 1, p\rangle\langle p, p^{-1}\rangle = {}^{f_0(1)}\langle p, p^{-1}\rangle\langle 1, 1\rangle,$$

so, as $f_0(1) = 1$ and $\langle 1, 1\rangle = 1$, we have $\langle 1, p\rangle = 1$.
Similarly, try $p_1 = 1$, $p_2 = p$ and $p_3 = p^{-1}$. ■

Remark: It will probably not have escaped your notice that what we have here is very closely related to a weak action of P on C'. This will become more apparent slightly later on.

We next look at the naturality of φ.
If we fix $p \in P$, we get the pre-whiskering

$$-\sharp_0 p : \mathcal{X}(\mathsf{C})(*, *) \to \mathcal{X}(\mathsf{C})(*, *),$$

and the corresponding post-whiskering

$$p\sharp_0- : \mathcal{X}(\mathsf{C})(*, *) \to \mathcal{X}(\mathsf{C})(*, *).$$

Naturality of φ means that pre- (resp. post-) whiskering in $\mathcal{X}(\mathsf{C})$ is translated into the similar operation in $\mathcal{X}(\mathsf{C}')$.

Pre-whiskering naturality: For any $p_1, p_2 \in P$ and $c \in C$, the diagram

$$\begin{tikzcd}
f_0(p_2p_1) \arrow[r, "\varphi_{p_2,p_1}"] \arrow[d, "F_2(c,p_2p_1)"'] & f_0(p_2)f_0(p_1) \arrow[d, "F_2(c,p_2)\sharp_0 f_0(p_1)"] \\
f_0(p_2'p_1) \arrow[r, "\varphi_{p_2',p_1}"'] & f_0(p_2')f_0(p_1)
\end{tikzcd}$$

in $\mathcal{X}(\mathsf{C}')$ commutes, where $p_2' = \partial c.p_2$.
Using F_2' and $\langle -, -\rangle$, this translates as

Lemma 36 ***(Primitive pre-whiskering condition.)*** *For $p_1, p_2 \in P$ and $c \in C$,*

$$\langle \partial c.p_2, p_1\rangle.F_2'(c, p_2p_1) = F_2'(c, p_2).\langle p_2, p_1\rangle.$$

■

We call it 'primitive' as we really want it in terms of f_1 not of F_2'.

Post-whiskering naturality: For any $p_2, p_3 \in P$ and $c \in C$, the diagram

$$\begin{tikzcd}
f_0(p_3p_2) \arrow[r, "\varphi_{p_3,p_2}"] \arrow[d, "F_2({}^{p_3}c, p_3p_2)"'] & f_0(p_3)f_0(p_2) \arrow[d, "f_0(p_3)\sharp_0 F_2(c,p_2)"] \\
f_0(p_3'p_2) \arrow[r, "\varphi_{p_3,p_2'}"'] & f_0(p_3)f_0(p_2')
\end{tikzcd}$$

in $\mathcal{X}(\mathsf{C}')$ commutes, where $p_2' = \partial c.p_2$.

Using F_2' and $\langle -, -\rangle$, this translates as

Lemma 37 ***(Primitive post-whiskering condition.)*** *For $p_2, p_3 \in P$ and $c \in C$,*

$$\langle p_3, \partial c.p_2\rangle.F_2'({}^{p_3}c, p_3p_2) = {}^{f(p_3)}F_2'(c, p_2).\langle p_3, p_2\rangle.$$

■

Recall that we wrote $f_1(c)$ for $F_2'(c, 1)$. Using naturality, and from the fact that an arbitrary (c, p) can be written as $(c, 1)\sharp_0(1, p)$, we can derive a rule expressing $F_2'(c, p)$ in terms of $f_1(c)$ and $\langle -, -\rangle$:

Lemma 38 *For any c, p, as above,*

$$F_2'(c, p) = \langle \partial c, p\rangle^{-1} f_1(c).$$

Proof: Pre-whiskering naturality gives

$$\langle \partial c, p\rangle.F_2'(c, p) = F_2'(c, 1).\langle 1, p\rangle,$$

but we showed that $\langle 1, p\rangle$ is the identity, so the result follows. ■

Of course, as F_2 is a functor, we also know that $f_1(1) = 1$.

It is thus possible to define $F_2(c, p)$ in terms of the pairing function $\langle -, -\rangle$ together with f_0 and f_1. Of course, we need to be sure that F_2, thus (re-)constructed, has the right properties, mainly as a check that the whole framework holds together, and that we have successfully reduced the data specifying F to a usefully presented description. For instance, $F_2(c, p)$ is to be a 2-cell from $f_0(p)$ to $f_0(\partial c.p)$, i.e., we must have:

Lemma 39 *Thus defined, $F_2'(c, p)$ satisfies $f_0(\partial c.p) = \partial F_2'(c, p).f_0(p)$.*

Proof: (Included really only because it is quite neat. It could have been left to you.)

$$\partial F_2'(c,p) = \partial\langle \partial c, p\rangle^{-1}\partial f_1(c),$$

but we know $\partial f_1(c) = f_0(\partial c)$. We obtain

$$\partial\langle \partial c, p\rangle = f_0(\partial c) f_0(p) f_0(\partial c.p)^{-1},$$

and hence

$$\partial\langle \partial c, p\rangle^{-1} = f_0(\partial c.p) f_0(p)^{-1} f_0(\partial c)^{-1},$$

so

$$\partial F_2'(c,p) = f_0(\partial c.p) f_0(p)^{-1},$$

or

$$f_0(\partial c.p) = \partial F_2'(c,p).f_0(p),$$

as required. ■

Proposition 48 ***Pre-whiskering naturality:*** *For $p_1, p_2 \in P$ and $c \in C$,*

$${}^{f_0(\partial c)}\langle p_2, p_1\rangle.f_1(c) = f_1(c).\langle p_2, p_1\rangle.$$

Proof: By calculation after substituting: on substituting $\langle \partial c, p\rangle^{-1} f_1(c)$ for $F_2'(c,p)$, etc., the primitive version gives

$$\langle \partial c.p_2, p_1\rangle.\langle \partial c, p_2.p_1\rangle^{-1} f_1(c) = \langle \partial c, p_2\rangle^{-1} f_1(c)\langle p_2, p_1\rangle.$$

By the associativity cocycle condition,

$$\langle \partial c.p_2, p_1\rangle.\langle \partial c, p_2.p_1\rangle^{-1} = \langle \partial c, p_2\rangle^{-1\,f_0(\partial c)}\langle p_2, p_1\rangle.$$

Cancellation of $\langle c, p_2\rangle^{-1}$ in the combined expression gives the result. ■

Remark: Rearranging the above equation gives

$${}^{\partial f_1(c)}\langle p_2, p_1\rangle = f_1(c)\langle p_2, p_1\rangle f_1(c)^{-1},$$

which is related to the Peiffer identity,

$${}^{\partial c}c' = c.c'c^{-1},$$

within C' and could have been deduced directly from it.

Back again, this time to Post-Whiskering Naturality, we had

$$\langle p_3, \partial c.p_2\rangle.F_2'({}^{p_3}c, p_3p_2) = {}^{f(p_3)}F_2'(c, p_2).\langle p_3, p_2\rangle,$$

and hence

$$\langle p_3, \partial c.p_2\rangle.\langle p_3\partial c.p_3^{-1}, p_3p_2\rangle^{-1} f_1({}^{p_3}c) = {}^{f_0(p_3)}\langle \partial c, p_2\rangle^{-1\,f_0(p_3)} f_1(c).\langle p_3, p_2\rangle.$$

Using the 'associativity' cocycle condition gives an expression for the first part of the right hand side as

$${}^{f_0(p_3)}\langle \partial c, p_2\rangle = \langle p_3, \partial c\rangle\langle p_3.\partial c, p_2\rangle\langle p_3, \partial c.p_2\rangle^{-1},$$

so we get, after an easy rearrangement:

Proposition 49 ***Post-whiskering naturality:*** *For $p_2, p_3 \in P$ and $c \in C$,*

$$\langle p_3.\partial c.p_3^{-1}, p_3p_2\rangle^{-1} f_1({}^{p_3}c) = \langle p_3.\partial c, p_2\rangle^{-1} \langle p_3, \partial c\rangle^{-1 f_0(p_3)} f_1(c).\langle p_3, p_2\rangle.$$

∎

Remarks: (i) This formula, or rather the right action / algebraic composition order form of it, is ascribed to Ettore Aldrovandi in the corrected version of Noohi's notes, [108]. It is worth noting that Noohi uses right actions and a lax functor formulation, so, for instance,

$$\varphi : F(b)F(a) \Rightarrow F(ba).$$

This results in there being no inverse on the pairing brackets, amongst other things.

(ii) If we consider the case $p_3 = p_2^{-1} = p$, say, then we get

$$f_1({}^pc) = \langle p.\partial c, p^{-1}\rangle^{-1} \langle p, \partial c\rangle^{-1 f_0(p)} f_1(c) \langle p, p^{-1}\rangle,$$

which is a form of Noohi's 'equivariance condition', cf. [108].

We can use similar arguments to these above to investigate f_1 further.

Proposition 50 *The map $f_1 : C \to C'$ satisfies: for all $c_2, c_1 \in C$,*

$$f_1(c_2c_1) = \langle \partial c_2, \partial c_1\rangle^{-1} f_1(c_2) f_1(c_1).$$

Proof: Using the definition of f_1,

$$\begin{aligned}
(f_1(c_2c_1), 1) &= (F_2(c_2c_1, 1) \\
&= F_2(c_2, \partial c_1) F_2(c_1, 1) \\
&= (\langle \partial c_2, \partial c_1\rangle^{-1} f_1(c_2), \partial c_1) \sharp_1 (f_1(c_1), 1) \\
&= (\langle \partial c_2, \partial c_1\rangle^{-1} f_1(c_2) f_1(c_1), 1)
\end{aligned}$$

as required. ∎

We thus have that f_1 is almost a homomorphism. It is 'deformed' by the term $\langle \partial c_2, \partial c_1\rangle$.

We could, as might be expected, derive this also from a combination of pre- and post-whiskering and the interchange law. As the interchange law holds in both $\mathcal{X}(\mathsf{C})$ and $\mathcal{X}(\mathsf{C}')$, and as F_2 is a functor, it must relate these two, preserving 'interchange'.

Suppose we have

$$\alpha : p_1 \Rightarrow p_1',$$
$$\beta : p_2 \Rightarrow p_2',$$

then we have a diagram,

$$\begin{tikzcd}
f_0(p_2p_1)) \arrow[r, Rightarrow, "F(\beta\sharp_0\alpha)"] \arrow[d, Rightarrow, "\varphi_{p_2,p_1}"'] & f_0(p_2'p_1') \arrow[d, Rightarrow, "\varphi_{p_2',p_1'}"] \\
f_0(p_2)f_0(p_1) \arrow[r, Rightarrow, "F(\beta)\sharp_0F(\alpha)"'] & f_0(p_2')f_0(p_1')
\end{tikzcd}$$

which will commute in $\mathcal{X}(\mathsf{C}')$.

We can translate this, as before, in terms of $\langle -, -\rangle$, f_0 and f_1.

Proposition 51 *For* $\alpha = (c_1, p_1)$ *and* $\beta = (c_2, p_2)$,

$$\langle \partial c_2.p_2, \partial c_1\rangle\langle \partial c_2.p_2\partial c_1 p_2^{-1}, p_2p_1\rangle^{-1} f_1({c_2}^{p_2}c_1) = \langle \partial c_2, p_2\rangle^{-1} f_1(c_2)^{f_0(p_1)}\langle \partial c_1, p_1\rangle^{-1}.{}^{f_0(p_1)}f_1(c_1)\langle p_2, p_1\rangle.$$

■

We **leave the proof to you**. The resulting formula reduces to the pre- and post- forms for suitable choices of the variables. In turn, it can be derived by algebraic manipulation from those forms together with the formula for $f_1(c_2c_1)$ in terms of $f_1(c_2)$ and $f_1(c_1)$. The added complexity of the interchange form makes its use less attractive than that of the reduced forms.

Analysing pseudo-functors between 2-groups has thus led us to a list of structure and related properties that we can extract to get the following algebraic form of the definition. As usual, C and C' are two crossed modules.

Definition: Weak map, algebraic form: A *weak map*, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, is given by the following structure:

- a function, $f_0 : P \to P'$;
- a function, $f_1 : C \to C'$;
- a pairing, $\langle\ \ ,\ \ \rangle : P \times P \to C'$.

These are to satisfy:

W1 (**Normalisation**): $f_0(1) = 1$ and $\langle 1, 1\rangle = 1$;

W2 (**'Almost a homomorphism' for** f_1): for $c_2, c_1 \in C$,

$$f_1(c_2c_1) = \langle \partial c_2, \partial c_1\rangle^{-1} f_1(c_2) f_1(c_1);$$

W3 (**'Almost a homomorphism' for** f_0): for $p_1, p_2 \in P$,

$$f_0(p_2p_1) = \partial\langle p_2, p_1\rangle^{-1} f_0(p_2) f_0(p_1);$$

W4 (**Cocycle**): for $p_1, p_2, p_3 \in P$,

$$\langle p_3, p_2\rangle.\langle p_3p_2, p_1\rangle = {}^{f_0(p_3)}\langle p_2, p_1\rangle.\langle p_3, p_2p_1\rangle;$$

W5 (**Whiskering conditions**):

Pre: for $p_1, p_2 \in P$ and $c \in C$,

$${}^{f_0(\partial c)}\langle p_2, p_1\rangle.f_1(c) = f_1(c).\langle p_2, p_1\rangle;$$

Post: for $p_2, p_3 \in P$ and $c \in C$,

$$\langle p_3.\partial c.p_3^{-1}, p_3p_2\rangle^{-1} f_1({}^{p_3}c) = \langle p_3.\partial c, p_2\rangle^{-1}\langle p_3, \partial c\rangle^{-1\, f_0(p_3)} f_1(c).\langle p_3, p_2\rangle.$$

We then have:

Theorem 11 *(Noohi, [108]) The two definitions of weak map, pseudo-functorial and algebraic, are equivalent.* ∎

Remarks: (i) The proof in one direction has been sketched out above, and some indication has been given as to how to go in the other direction. The details of that direction are a **'good exercise for the reader'**.

(ii) In the published form (that is in [107]), the additional assumption that f_1 was a homomorphism was made. This is not a consequence of the pseudo-functorial definition of a weak map. A correction was made available by Noohi, in [108], where the axioms are given in more or less the above form with, however, right actions, etc.

(iii) It should be noted that we have not encoded weak / pseudo- natural transformations in the above. In [108], there is a description of such things within the context of the algebraic definition of weak maps as above. The task of translating that to the notational conventions used here is **left to you**.

(iv) Any morphism of crossed modules gives a weak map between them, with a trivial pairing function, and any weak map with trivial pairing likewise *is* a morphism of crossed modules. With morphisms of crossed modules composition is very easy to do, so what about composition of weak maps? This is again **left as an exercise** for you to investigate. We will shortly see the simplicial description of weak maps and in that description composition is just composition of simplicial maps, so is easy. As a consequence, as yet, no use for a composition formula in the algebraic form of the definition seems to have been found and we will not discuss it further, except to point out that to investigate it yourself can be a useful exercise in linking the 2-group(oid) way of thinking to the crossed module way.

(v) The above algebraic definition is not intended to be in a neatest form. Some of the conditions may be redundant, for instance. The list is inspired both by Noohi's notes, and the form given there, but also by the interpretation of each condition in terms of the pseudo-functorial one.

We observed earlier the similarity between the rules for a weak map, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, and those for a weak action. To clarify this a bit further, note that if $\mathsf{C} = (1, P, 1)$ is 'really a group', then a weak map, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, consists just of f_0 and φ, as the only value $f_1(c)$ can take is 1 corresponding to $c = 1$! It is a normalised pseudo-functor from $P[1]$ to $\mathcal{X}(\mathsf{C}')$.

A weak action of P on P' would be a pseudo-functor from $P[1]$ to $\mathsf{Aut}(P')$. The only difference between the two notions is to replace the automorphism 2-group, $\mathsf{Aut}(P')$ by the general 2-group, $\mathcal{X}(\mathsf{C}')$. A weak action of P on P' can thus be thought of as a weak map from P to $\mathsf{Aut}(P')$, (with allowance being made for a deliberate confusion between the 2-group of automorphisms of P' and the corresponding crossed module).

A natural generalisation of weak action of a group is thus a weak action of a crossed module, C, which can be defined to be an op-lax functor from $\mathcal{X}(\mathsf{C})$ to whatever 2-category you like. Equally well, you can make C act weakly on some object in a simplicially enriched setting by using an $\mathcal{S}$-functor from the corresponding simplicial group.

Finally we note the following very interesting and useful result.

Weak maps induce morphisms on homotopy groups.

More precisely,

Proposition 52 *Suppose that* $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ *is a weak map of crossed modules, then* f *induces morphisms*

$$\pi_i(\mathsf{f}) : \pi_i(\mathsf{C}) \to \pi_i(\mathsf{D})$$

for $i = 0, 1$.

Proof: There are several different proofs of this. Starting from the algebraic description, we have that f_0 induces a homomorphism from $P/\partial C$ to $P'/\partial C'$. (This looks to be 'immediate' from condition $W3$, but, of course, you do have to check that the apparently induced morphism is 'well-defined'. This is easy since $f_0(\partial c) = \partial f_1(c)$.) That handles the $i = 0$ case.

Suppose next that $c \in Ker\,\partial$, then clearly $f_1(c) \in Ker\,\partial'$. Is the resulting induced mapping a homomorphism? Of course, this follows from $W2$, and we are finished. ∎

There are also easy proofs of this coming from the simplicial description, as we will see.

We have already commented on the link between weak actions and maps between nerves / classifying spaces, and also on the links between extensions, sections and weak actions. We will shortly explore the extension of these links to give us more insight into weak maps.

5.7.2 The simplicial description

Suppose C and D are two crossed modules and $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ a weak map between them in the sense of the definition on page 195. We will rewrite this in a more 'pseudo-functorial' form as a pseudo-functor, $\mathcal{F} = (F, \gamma) : \mathcal{X}(\mathsf{C}) \to \mathcal{X}(\mathsf{D})$, between the corresponding 2-groupoids. By the properties of the nerve construction that we saw earlier in Proposition 44, there is equivalently a simplicial map,

$$f : Ner(\mathcal{X}(\mathsf{C})) \to Ner(\mathcal{X}(\mathsf{D})).$$

In this description, composition of weak maps is no problem, just compose the corresponding simplicial maps. Using the natural isomorphism from Proposition 47, from such an f, we get a corresponding morphism of (reduced) simplicial sets,

$$f : \overline{W}(K(\mathsf{C})) \to \overline{W}(K(\mathsf{D})),$$

and, by the adjunction between $\overline{W}$ and the loop groupoid functor, G, (mentioned back in section **??**, page **??**), we get a morphism of simplicial *groups*,

$$\overline{f} : G\overline{W}(K(\mathsf{C})) \to K(\mathsf{D}).$$

The simplicial group, $K(\mathsf{D})$, has a Moore complex of length 1, so $\overline{f}$ factors via a quotient of $G := G\overline{W}(K(\mathsf{C}))$, giving K of the crossed module $M(G, 1)$, i.e., the Moore complex of this quotient will be the crossed module:

$$\partial : \frac{NG_1}{d_0(NG_2)} \to G_0.$$

As G is a free simplicial group, this will have G_0 a free group.

There is a morphism, $G \to K(\mathsf{C})$, corresponding to the identity morphism from $\overline{W}(K(\mathsf{C}))$ to itself, so this is the counit of the adjunction and is a weak equivalence of simplicial groups, i.e., it induces isomorphisms on all homotopy groups. We thus get a span

$$K(\mathsf{C}) \overset{\varepsilon_{K(\mathsf{C})}}{\longleftarrow} G \longrightarrow K(\mathsf{D}),$$

or, passing to crossed modules,

$$\mathsf{C} \leftarrow M(G,1) \rightarrow \mathsf{D}.$$

We know that the left hand part of the span is a weak equivalence of crossed modules in the sense of section 3.1 (or of simplicial groups, if we go back a line or two), so what really is this G? It was formed from $\overline{W}(K(\mathsf{C}))$ by applying the loop groupoid functor, G, which is left adjoint to $\overline{W}$ and, as we said above, the natural map, $G\overline{W} \rightarrow Id$ is the counit of that adjunction. The results that we mentioned earlier (due to Dwyer and Kan, [59], or originally, as we really are only looking at the reduced case, to Kan, [82]) include that this is a weak equivalence, i.e., it induces isomorphisms on all homotopy groups. (Look up the theory in Goerss and Jardine, [66], for example, if you need more detail.)

This observation gives us a second proof of the result from page 203.

Proposition 53 *(Simplicial version of Proposition 52) Suppose that* $\mathsf{f} : \mathsf{C} \rightarrow \mathsf{D}$ *is a weak map of crossed modules, then* f *induces morphisms,*

$$\pi_i(\mathsf{f}) : \pi_i(\mathsf{C}) \rightarrow \pi_i(\mathsf{D}),$$

for $i = 0, 1$.

Simplicial Proof: We consider f as the span,

$$\mathsf{C} \leftarrow M(G,1) \rightarrow \mathsf{D}.$$

Now applying π_i, we get

$$\pi_i(\mathsf{C}) \overset{\cong}{\leftarrow} \pi_i(M(G,1)) \rightarrow \pi_i(\mathsf{D}),$$

but the left hand side is a natural isomorphism, and the induced morphism is the composite of that isomorphism's inverse followed by the induced morphism coming from the right hand branch of the span. ■

We still need to describe G in any detail, and to do this we need to revisit the loop groupoid functor, $G(-)$, and, as we have used the conjugate $\overline{W}$, we must take its conjugate, i.e., the functional composition order version of that construction.

5.7.3 The conjugate loop groupoid

It will be convenient to present the conjugate version of the Dwyer-Kan loop groupoid, that is the one that corresponds to the functional composition order and to the form of $\overline{W}$ that we have just seen, above page 190. The precise description, once we have it, will have an obvious relation with the more standard form that we have seen earlier (page **??**), but we will take the opportunity to explore a little why this works and so will pretend to forget that we have seen the other form.

We suppose given a simplicial map, $f : K \rightarrow \overline{W}H$ for H an $\mathcal{S}$-groupoid, where we take $\overline{W}$ in the 'functional' form above, (page 190). We want to construct an 'adjoint map', $\overline{f} : G(K) \rightarrow H$, but as yet do not have an explicit description of G.

We have $G(K)$ will be some $\mathcal{S}$-groupoid on the object set, K_0, and $\overline{f}$ on objects will just be f_0 (on vertices). We know $G(K)_0$ will be some groupoid and $\overline{f}$, on an arrow $g : x \rightarrow y$, must be determined by f_1 on K_1, so the obvious solution is that $G(K)_0$ will be the free groupoid on the non-degenerate 1-simplices. (We must put $s_0(x) = id_x$, for $x \in K_0$. That is needed to get identities to work correctly - for **you to investigate**.) We will use functional composition order in $G(K)_0$, of course.

Defining, for $x \in K_0$, $\overline{x}$ to denote the corresponding object of $G(K)$, then, for $k \in K_1$, we will extend the overline notation and write $\overline{k} : \overline{d_1 k} \to \overline{d_0 k}$ for the corresponding generator of $G(K)_0$ and then $\overline{f}(\overline{k})_0 : f_0 d_1(k) \to f_0 d_0(k)$ in H_0, will be given by $f_1(k)$. (Freeness of $G(K)_0$ guarantees that this $\overline{f}_0$ exists and is unique with the correct universal property.)

The fun starts in dimension 1. Suppose now $k \in K_1$, then

$$f_2(k) = (h_2, h_1) \in \overline{W}(H)_2,$$

and we will write $h_2 = h_2(k)$, $h_1 = h_1(k)$, as these simplices clearly depend on the input k. We have $h_i(k) \in H_{i-1}$ and $s(h_2(k)) = t(h_1(k))$.

We need a groupoid, $G(K)_1$ with K_0 as its set of objects, and a map $\overline{f}_1 : G(K)_1 \to H_1$. (We expect 'freeness' as we have a left adjoint - but free on what? There are several choices to try and several of them work, since we are in a groupoid and, to some extent, we are making a *choice* of generators, so conjugate generators might also give a valid choice and an isomorphic $G(K)_1$.) Writing $\overline{k}$ for the generator corresponding to $k \in K_2$, we do not know what the source and target of $\overline{k}$ should be. Clearly they have to be amongst its vertices! Which ones? There are three of them!

Rather than choose the obvious one with source being the vertex of k corresponding to 0 (i.e., $d_1 d_2(k)$) and target being that corresponding to 2 (so $d_0 d_0(k)$), we will look at $\overline{f}$ and see if there are advantages with any other choice. Looking at $\overline{f}(\overline{k})_1$, it has to be in H_1 and we already have an element of that groupoid namely $h_2(k)$. This suggests that we try *defining* $\overline{f}(\overline{k})_1$ to be $h_2(k)$ and see what that implies for $\overline{k}$ itself.

We have

$$\begin{aligned} f(d_0(k)) &= d_0(f(k)) = (d_0 h_2(k)), \\ f(d_1(k)) &= d_1(f(k)) = (d_1 h_2(k).h_1(k)), \\ f(d_2(k)) &= d_2(f(k)) = (h_1(k)), \end{aligned}$$

and, if we take

$$\overline{f}_1(\overline{k}) = h_2(k),$$

then

$$\begin{aligned} d_0 \overline{f}_1(\overline{k}) &= d_0 h_2(k) = f(d_0(k)), \\ d_1 \overline{f}_1(\overline{k}) &= d_1 h_2(k) = f(d_1(k)).f(d_2(k))^{-1}, \end{aligned}$$

so as to cancel the $h_1(k)$ term. This suggests that we define $d_0(\overline{k}) = \overline{d_0(k)}$, but $d_1(\overline{k}) = \overline{d_1(k)}(\overline{d_2(k)})^{-1}$. This corresponds to the source of $\overline{k}$ being the target of $\overline{d_2(k)}$, that is the object $\overline{d_0 d_2(k)} = \overline{d_1 d_0(k)}$, whilst the target of $\overline{k}$ would be the same as that of $\overline{d_0(k)}$, namely the object $\overline{d_0 d_0(k)}$.

Those are the natural choices for *that* choice of $\overline{f_1}$. To summarise

- if $k \in K_2$, $s(\overline{k}) = \overline{d_1 d_0(k)}$, $t(\overline{k}) = \overline{d_0^{(2)}(k)}$, whilst
  - $d_0(\overline{k}) = \overline{d_0(k)}$,
  - $d_1(\overline{k}) = \overline{d_1(k)}(\overline{d_2(k)})^{-1}$,

and it works.

We define $s_i(\overline{k}) = \overline{s_i(k)}$ for $0 \leq i \leq n-1$ and set $\overline{s_n(k)} =$ identity, and do this for all n, although we have not yet looked at $k \in K_n$ for $n > 2$, to which we turn next:

- For $k \in K_n$, in general, we take $\overline{k} \in G(K)_{n-1}$ with
  - $s(\overline{k}) = \overline{d_1 d_0^{(n-1)}(k)}$,
  - $t(\overline{k}) = \overline{d_0^{(n)}(k)}$

  with $G(K)_{n-1}$ free on the graph,

$$K_n \underset{t}{\overset{s}{\rightrightarrows}} K_0 \ ,$$

excepting the edges $s_n(x)$ for $x \in K_{n-1}$.

The face maps are given by

  - $d_i(\overline{k}) = \overline{d_i(k)}$ for $0 \leq i < n-1$,
  - $d_{n-1}(\overline{k}) = \overline{d_{n-1}(k)}(\overline{d_n(k)})^{-1}$.

It is easy to check that these satisfy the simplicial identities with the degeneracies as given earlier.

We have chosen this source and target, based on a reasonable choice for $\overline{f}$, but there are other choices that could perhaps have been made. For instance, for $(h_2, h_1) \in \overline{W}(H)_2$ with $s(h_2) = t(h_1)$, but that, perhaps, suggests forming $h_2 \cdot s_1(h_1)$, or similar, and this might give another way of defining generators for $G(K)_{n-1}$ and hence a different expression for the elements. We would expect that the result is isomorphic to the G that we have written down, as both *should* be adjoint to $\overline{W}$. The inconvenience of the definition that we have given is that the source and target of $\overline{k}$ seem very strange. It would be nice to have, for instance, for $k \in K_2$, $s(\overline{k}) = \overline{d_1 d_2(k)}$ and $t(\overline{k}) = \overline{d_0 d_0(k)}$ as these, naively, look to be where the simplex starts and ends. Such a choice would make it easier to link it with the left adjoint of the homotopy coherent nerve functor. On the 'plus side', for the G that we have written down (and also for the Dwyer - Kan original version), is that it has an easy unit and counit for the adjunction and a clear link with the twisting function (cf. page **??**) for the reduced case. (The other choices suggested may also work *and* the links with twisting function formulations of twisted cartesian products may be as clear in that revised form. (I have never seen it explored. Such an exploration would be a **good exercise to do**. If it works well, it could be useful; if it does not work out, why not? **Perhaps some reader will attempt this.** I do not know the answer.)

We have stated that this form of G is left adjoint to the 'functional form' of $\overline{W}$ and we launched into this to examine what the idea of 'weak morphism' would give at the 'elementwise' level. Remember, a weak morphism from C to D corresponded to a map of simplicial groups from $G\overline{W}(K(\mathsf{C}))$ to $K(\mathsf{D})$. The counit of the adjunction goes from $G\overline{W}$ to Id and one way to get some data that correspond to a weak morphism is to find some neat way of describing a section of this from $K(\mathsf{C})$ to $G\overline{W}(K(\mathsf{C}))$. That would, we may suppose, correspond to a weak morphism from C to $M(G,1)$, where $G = G\overline{W}(K(\mathsf{C}))$.)

For this to be feasible, we need to know more about the counit, $\varepsilon : G\overline{W}(H) \to H$, in general, and so may as well look at the unit, $\eta : K \to \overline{W}G(K)$, as well, so as to indicate the structures behind this adjunction.

The unit, $\eta_K : K \to \overline{W}G(K)$: Remember what η_K is. It corresponds, in the adjunction, to the identity on $G(K)$, so one way to derive the following formulae is to work out $\underline{f} : K \to \overline{W}(H)$, when starting with $f : G(K) \to H$.

We have that if $k \in K_n$, then $\underline{f}_n(k)$ will be of the form $(h_n, \ldots, h_1)$ with $h_i \in H_{i-1}$, as before. Looking at $d_n(\underline{f}_n(k) = \underline{f}_{n-1}(d_n(k))$ gives us $(h_{n-1}, \ldots, h_1)$ and allows us to use induction to get all but $h_n \in H_{n-1}$, but we also have that $f_{n-1}(k) \in H_{n-1}$, so we have an obvious candidate for that missing element.

You can easily follow through this process, either for a general $f : G(K) \to H$, or just for $f : G(K) \to G(K)$ being the identity morphism, and this gives η_K.

To write η_K down neatly, it is useful to introduce an abbreviation. If $k \in K_n$, its *last* listed face is d_nk and we will need to iterate this last face construction, $d_{n-1}d_n(k)$ and so on. Rather than have long strings $d_i \ldots d_{n-1}d_n(k)$, we will write 'L' for 'last' and so define

$$d_L^{(m)} = d_{n-m+1} \ldots d_{n-1}d_n$$

as the m-iterated last face operator. With this notation, for $k \in K_n$,

$$\eta_K k = (\overline{k}, \overline{d_n(k)}, \ldots, \overline{d_L^{(n-1)}(k)}).$$

(You are **left to check the detail**.)

The counit, $\varepsilon_H : G\overline{W}(H) \to H$: We have already seen how to build $\overline{f} : G(K) \to H$ if we start with $f : K \to \overline{W}(H)$, as that was how we sorted out the structure in this version of $G(K)$. Given such an f, where $f_n(k) = (h_n(k), \ldots, h_1(k))$, we had that

$$\overline{f}_{n-1}(\overline{k}) = h_n(k).$$

We thus get, in particular, that if we have $\underline{h} = (h_n, \ldots, h_1)$ in $\overline{W}(H)$, then

$$\varepsilon_H(\overline{\underline{h}}) = h_n,$$

so is almost a 'projection' defined on the generators. (Of course, it resembles even more the counit of the free group(oid) monad which evaluates a word in the elements of a group.)

5.7.4 Identifying $M(G, 1)$

It is not difficult to *start* identifying the Moore complex, $N(G\overline{W}(H))$, in terms of free groups on Moore complex terms from H itself. You can do this with 'bare hands' and it is quite instructive. A complete verification of what you might suspect the terms to be is quite tricky, however, so we will limit ourselves to the case $H = K(\mathsf{C})$ for C, our 'usual' crossed module, $\mathsf{C} = (C, P, \partial)$, as, there, $N(K(\mathsf{C}))_n$ is trivial for $n \geq 2$, and we will even avoid calculating $N(G\overline{W}(K(\mathsf{C})))_1$, as we really need its quotient $M(G\overline{W}(K(\mathsf{C})), 1)$. (We will, as before, write G for $G\overline{W}(K(\mathsf{C}))$, for convenience.)

We will use a neat argument to identify the crossed module, $M(G\overline{W}(K(\mathsf{C})), 1)$, via another route. Before that we *will* look at the bottom terms of the Moore complex of this G.

We write $\underline{h} = (h_n, \ldots, h_1)$, so this defines a generator $\overline{\underline{h}}$ in G_{n-1}. We thus have G_0 is freely generated by the elements of P, i.e., $G_0 \cong FU(P)$, where F is the free group functor and U the underlying set functor.

We can examine a generator, $\overline{\underline{h}}$, for $n = 2$, i.e., in G_1, and

$$d_1(\overline{\underline{h}}) = \overline{d_1(\underline{h})}.\overline{d_2(\underline{h})}^{-1} = \overline{(d_1(h_2).h_1)}.\overline{(h_1)}^{-1}.$$

We immediately can see that such a term will vanish if $d_1(h_2)$ is trivial and with a little more work can show that a word in such terms and their inverses vanishes if d_1 of the h_2-parts of it vanishes. (We will leave this slightly vague as the calculation is **worth doing** and this **is worth pursuing** on your own, so as to get a better 'elementary' understanding of G_1 - and, in fact, of higher G_n in more generality.) This suggests that $N(G)_1$ may be the free group on the underlying set of $NK(\mathsf{C})_1$, but does not by itself prove this (and as we will side-step this calculation shortly, we do not need to do it now).

Of course, $M(G,1)$ has 'top term' $NG_1/d_0(NG_2)$, so attacking at the elementwise level, the next step would seem to be to work out NG_2 or rather $d_0(NG_2)$ as that is all we need for the moment. We will not, in fact, do this, although, we repeat, it is **worthwhile doing so**, instead we will backtrack a little and review the problem from another direction, one that we visited a few pages back.

We have the counit of the adjunction, giving

$$\varepsilon : G \to K(\mathsf{C}),$$

and, by the construction of the associated crossed complex, $\mathsf{C}(G)$, of the simplicial group G, an adjoint induced map,

$$\mathsf{C}(G) \to \mathsf{C}.$$

This factorises via the map

$$M(G,1) \to \mathsf{C},$$

that we are seeking to understand. For this last step, we are using that $M(G,1)$ is left adjoint to the natural inclusion of the category of crossed modules into that of crossed complexes (both can be 'reduced' or unreduced, it makes no difference).

We also had that $\mathsf{C}(-)$ was left adjoint to the 'inclusion' of crossed complexes (disguised, via K and the Dold-Kan theorem, as group (or groupoid) T-complexes) into all simplicial groups (or $\mathcal{S}$-groupoids). This chain of left adjoints translates into a single universal property, one which is very useful.

If we have any crossed module E having $FU(P)$ at its base, and any morphism

$$\mathsf{f} : \mathsf{E} \to \mathsf{C},$$

having that $f_0 : E_0 \to P$ is $\varepsilon_P : FU(P) \to P$, the counit of the free group monad, then we can factor f through the pullback crossed module, $\varepsilon_P^*(\mathsf{C})$:

$$\begin{tikzcd} E_1 \arrow[rr] \arrow[dr] & & E_1 \times_P C \arrow[dl] \\ & E_0 & \end{tikzcd}$$

(see page 37 and note that here $\varepsilon_P^*(\mathsf{C})_1 \cong E_1 \times_P C$). We will generalise this slightly in a moment, but first we introduce some terminology. As before, $\mathsf{C} = (C, P, \partial_{\mathsf{C}})$ and $\mathsf{D} = (D, Q, \partial_{\mathsf{D}})$ are crossed modules:

Definition: (i) A map, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, of crossed modules is a *fibration* if $f_1 : C \to D$ and $f_0 : P \to Q$ are both epimorphisms of groups.

(ii) A map, f, as above, is a *trivial fibration* if it is a fibration and the induced map,

$$C \to D \times_Q P,$$

is an isomorphism.

Remarks: (i) If $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ is a fibration, it should be obvious that $K(\mathsf{f}) : K(\mathsf{C}) \to K(\mathsf{D})$ is a dimensionwise epimorphism of simplicial groups and hence is a fibration of such (in the sense we discussed in section 1.3.5, page 30). We therefore get a fibration exact sequence of homotopy groups. We set $\mathsf{B} = Ker(\mathsf{f})$, that is, $(Ker(f_1), Ker(f_0), \partial)$ for the restricted $\partial = \partial_{\mathsf{C}}|_{Ker(f_1)}$, and then obtain

$$1 \to \pi_1(\mathsf{B}) \to \pi_1(\mathsf{C}) \to \pi_1(\mathsf{D}) \to \pi_0(\mathsf{B}) \to \pi_0(\mathsf{C}) \to \pi_0(\mathsf{D}) \to 1.$$

This is just the usual $Ker - Coker$ 6-term exact sequence of homological algebra, but in a slightly non-Abelian context.

(**Remember** that in our notation $\pi_1(\mathsf{C}) = Ker\,\partial_{\mathsf{C}}$ and $\pi_0(\mathsf{C}) = Coker\,\partial_{\mathsf{C}} \cong P/\partial_{\mathsf{C}}C$. This is a shift of index from the notation used in some sources, where our $\pi_1(\mathsf{C})$ would be their $\pi_2(\mathsf{C})$, because it is the π_2 of the classifying space of C. Likewise our π_0 is their π_1, so **always check** when comparing results.)

(ii) Suppose now that

$$\begin{tikzcd} C \arrow[r, "f_1"] \arrow[d, "\partial_{\mathsf{C}}"'] & D \arrow[d, "\partial_{\mathsf{D}}"] \\ P \arrow[r, "f_0"'] & Q \end{tikzcd}$$

is a pullback square (which is just saying that $C \to D \times_Q P$ is an isomorphism). It is well known that that implies that the kernels of ∂_{C} and ∂_{D} are isomorphic (via the restricted f_1). That fact is general and has a useful, easy categorical proof, but, none-the-less, we will give an 'element-wise' one, since it shows different aspects that can also be useful. It is equally easy, but slightly less general.

We replace C by $D \times_Q P$, so an element of this is a pair, (d, p), such that $\partial_{\mathsf{D}} d = f_0 p$. The description of ∂_{C} is then $\partial_{\mathsf{C}}(d, p) = p$, the second projection morphism. If $(d, p) \in Ker\,\partial_{\mathsf{C}}$, then $p = 1_P$ and $\partial_{\mathsf{D}} d = f_0 1_P = 1_Q$, so the isomorphism claimed associates $(d, 1)$ and d, where $d \in Ker\,\partial_{\mathsf{D}}$.

Going back to the exact sequence, we have that the induced map from $\pi_1(\mathsf{C})$ to $\pi_1(\mathsf{D})$ is an isomorphism in this case (as $\pi_1(\mathsf{B})$ is trivial). We can calculate B explicitly, of course. Identifying C with $D \times_Q P$ once again, $f_1(d, p) = d$, so $(d, p) \in Ker\, f_1$ if $d = 1_D$, and then, of course, $f_0(p) = 1_Q$, so $p \in Ker\, f_0$. The crossed module, B, is thus isomorphic to the crossed module, $(Ker\, f_0, Ker\, f_0, id)$, so, again of course, $\pi_1(\mathsf{B})$ is trivial! It is then clear that $\pi_0(\mathsf{B})$ is also trivial. In other words,

Lemma 40 *A trivial fibration of crossed modules is a weak equivalence.* ∎

The particularly useful case of this is the following: Given a crossed module, $\mathsf{C} = (C, P, \partial_{\mathsf{C}})$, pick a free group F together with an epimorphism,

$$\varepsilon : F \to P,$$

(for instance, if given a presentation of P, use the free group on the given set of generators). Form $\varepsilon^*(\mathsf{C}) = (C \times_P F, F, \partial')$, which will be, as we know, a crossed module. There is an induced fibration,

$$\mathsf{f} : \varepsilon^*(\mathsf{C}) \to \mathsf{C},$$

and this will be, by construction, a trivial fibration.

Example: We could take $F = FU(P)$, the free group on the underlying set of P with the counit, ε_P, as the epimorphism. Our earlier discussion suggested that $\varepsilon_P^*(\mathsf{C})$ looks somewhat like our G from there.

This example is what we will need, but it is not the only one around, of course. (That 'looks somewhat like' is vague and we need to do better than that! Here is that in detail.)

Proposition 54 *For $\mathsf{C} = (C, P, \partial_{\mathsf{C}})$, $G = G\overline{W}(K(\mathsf{C}))$, and $\varepsilon_P : FU(P) \to P$, as before, there is an isomorphism,*

$$M(G, 1) \cong \varepsilon_P^*(\mathsf{C}).$$

Proof: We know the base groups are both isomorphic to $FU(P)$, and so have to produce an isomorphism,

$$M(G, 1)_1 \cong FU(P) \times_P C,$$

over P, compatibly with the actions.

We certainly have the counit morphism,

$$\begin{tikzcd} M(G,1)_1 \arrow[r] \arrow[d, "\partial"'] & C \arrow[d, "\partial_C"] \\ FU(P) \arrow[r, "\varepsilon_P"'] & P \end{tikzcd}$$

which we will call f, for convenience. We know it is a weak equivalence, since $G\overline{W}(K(\mathsf{C})) \to K(\mathsf{C})$ is a weak equivalence of simplicial groups, so $Ker\,\mathsf{f}$ has trivial homotopy.

We get $\overline{f} : M(G, 1)_1 \to FU(P) \times_P C$ by the universal property of pullbacks. Explicitly

$$\overline{f}(h) = (\partial h, f_1(h)).$$

This map, $\overline{f}$, is a morphism of crossed modules by simple general arguments, (i.e., nothing to do with our particular situation here). We thus want to prove $\overline{f}$ is an isomorphism.

We note that $Ker\,\overline{f} \subseteq Ker\,f_1 \cap Ker\,\partial$, but $Ker\,\mathsf{f}$ has trivial π_1, so $Ker\,\overline{f}$ must be trivial and $\overline{f}$ is a monomorphism.

Is $\overline{f}$ an epimorphism? If $(h_0, c_1) \in FU(P) \times_P C$, so $f_0(h_0) = \partial c_1$, then pick $h_1 \in M(G, 1)_1$ such that $f_1(h_1) = c_1$, (**check** that f_1 is onto). We have

$$f_0(h_0) = f_0(\partial h_1),$$

so $h_0 = \partial h_1.k_0$ for some $k_0 \in Ker\,f_0$. We also have $\pi_0(Ker\,\mathsf{f})$ is trivial, so there is some $k_1 \in Ker\,f_1$ with $\partial k_1 = k_0$, but then $h' = h_1 k_1$ satisfies

$$\partial h' = h_0, \quad f_1(h') = f_1(h_1) = c_1,$$

so $\overline{f}$ is onto. ■

5.7.5 Cofibrant replacements for crossed modules

In other words, we have identified $M(G,1)$ completely and it has an easy description.

What about the properties of other $\varepsilon^*(\mathsf{C})$ for $\varepsilon : F \to P$, with F free? For the moment, this is prompted by curiosity, but it does provide some useful insights later on.

Our present situation is that a weak map from C to D is given by an actual map of crossed modules,

$$\varepsilon_P^*(\mathsf{C}) \to \mathsf{D},$$

and we also know that the map, $\varepsilon_P^*(\mathsf{C}) \stackrel{\simeq}{\to} \mathsf{C}$, is 'really' a counit or 'augmentation' of a resolution. We get a span

$$\mathsf{C} \stackrel{\simeq}{\leftarrow} \varepsilon_P^*(\mathsf{C}) \to \mathsf{D}.$$

What about other similar spans,

$$\mathsf{C} \stackrel{\simeq}{\leftarrow} \varepsilon^*(\mathsf{C}) \to \mathsf{D},$$

with ε, an epimorphism, $\varepsilon : F \to P$, and F a free group? Do they also give weak maps in some way? Of course, this is almost the same question as the previous one.

Before looking at this, we note a nice result:

Proposition 55 *For $\mathsf{C} = (C, P, \partial_{\mathsf{C}})$, with P a free group, the natural morphism $\varepsilon_P^*(\mathsf{C}) \stackrel{\simeq}{\to} \mathsf{C}$ is a split epimorphism.*

Proof: Of course, $\varepsilon : FU(P) \to P$ is split, since P is free. Let $\sigma_0 : P \to FU(P)$ be a splitting. From σ_0, we can construct

$$\sigma_1 : C \to FU(P) \times_P C,$$

by

$$\sigma_1(c) = (\sigma_0 \partial_{\mathsf{C}}(c), c),$$

as being the unique group homomorphism given by the pullback property. It is easy to check that (σ_1, σ_0) defines a crossed module morphism splitting the epimorphism induced by ε_P^*. ■

In fact, this split epimorphism is a trivial fibration, but we will not need this.

We next introduce a bit more of the homotopical terminology as applied to crossed modules, or equivalently to 2-group(oid)s. The ideas are derived from the paper, [100], by Moerdijk and Svensson. We first extend 'fibration' and 'trivial fibration' from crossed modules to 2-group(oid)s via the usual equivalence of categories. We give this in two forms, the first is from Noohi's paper, [108], the second from [100].

Definition: A morphism, $\psi : \mathcal{A} \to \mathcal{B}$, of 2-groupoids is called a *Grothendieck fibration* (or more simply a *fibration*) if it satisfies the following properties:

Fib. 1: for every arrow $b : B_0 \to B_1$ in $\mathcal{B}$ and every object, A_1, in $\mathcal{A}$ over B_1, (so $\psi(A_1) = B_1$), there is a lift $a : A_0 \to A_1$ with codomain, a_1;

Fib. 2: for every 2-arrow, $\beta : b_0 \Rightarrow b_1$ in $\mathcal{B}$, and every arrow a_1 in $\mathcal{A}$ such that $\psi(a_1) = b_1$, there is an arrow a_0 and a 2-arrow $\alpha : a_0 \Rightarrow a_1$ such that $\psi(\alpha) = \beta$.

The fibration is *trivial* if it is also a weak equivalence, i.e., inducing isomorphisms on π_0, π_1 and π_2.

Remark: This is nice as the first condition is a lifting condition for 1-arrows, whilst the second is one for 2-arrows. It is worth noting a slight more or less inconsequential choice is being made here. In covering space theory, it is usual to mention 'unique path lifting'. Recall that this relates to a continuous map of spaces, say $p : Y \to X$, and it requires that if $\lambda : I \to X$ is a path in X and we specify a point y_0 over $x_0 = \lambda(0)$, the starting point of λ, then there is a (unique) lift, $\tilde{\lambda}$, of λ starting at y_0.

In the above definition of fibration for 2-groupoids, no uniqueness is required, but also the specified point is the codomain of the 1-arrow, which intuitively corresponds to the end of the path rather than the start. This does not matter here as in a 2-groupoid both 1- and 2-arrows are invertible, but it is another instance of the lax / op-lax / pseudo 'conflict', so is worth noting that a choice *has* been made here.

Warning about the notation in 'trivial fibration': At the risk of repeating this too often, it should be noted that, if thinking of crossed modules rather than 2-groupoids, the above π_1 is the cokernel of the structure map and π_2 is its kernel. The set of connected components for a *2-group* will be a singleton. The π_1 of the 2-group is the π_0, in our notation, of the corresponding crossed module or simplicial group, and so on.

The alternative definition combines the two conditions in one. It occurs in Moerdijk and Svensson's paper, [100], so will be referred to as the M-S form of the definition.

Definition (alternative M-S form): A morphism, $\psi : \mathcal{A} \to \mathcal{B}$, of 2-groupoids is called a *Grothendieck fibration* (or more simply a *fibration*) if it satisfies the following condition:

for any arrow, $a : A_1 \to A_2$, in $\mathcal{A}$ and any arrows, $b_1 : B_0 \to \psi(A_1)$ and $b_2 : B_0 \to \psi(A_2)$, then any 2-arrow, $\alpha : b_2 \Rightarrow \psi(a) \circ b_1$, can be lifted to a 2-arrow, $\tilde{\alpha} : \tilde{b_2} \Rightarrow a \circ \tilde{b_1}$, (so $\psi(\tilde{\alpha}) = \alpha$, etc.).

Proposition 56 *The two forms of the definition are equivalent.*

Proof: We limit ourselves to a sketch, as the proof is quite easy, once you see that doing a fairly obvious thing is exactly what is needed. (Of course, the details are the **left to you as an exercise**.)

First assume we have a morphism satisfying the alternative (M-S) form of the definition. We must show it to have a lifting property for both 1- and 2-arrows.

Suppose we have $b : B_0 \to B_1$ in $\mathcal{B}$ and an object, A_1, in $\mathcal{A}$ over B_1, (so $\psi(A_1) = B_1$), then, in the alternative form, take $b_1 = b_2 = b$ with $\beta : b_1 \Rightarrow b_1$ the identity 2-arrow. The lift given by the M-S condition gives us a $\tilde{b} : A_0 \to A_1$ (and a $\tilde{\beta}$ that we do not actually need or use).

We thus have: 'M-S' $\Rightarrow$ '1-arrow lifting'.

To derive '2-arrow lifting' from 'M-S', we start with $\beta : b_0 \Rightarrow b_1$ and a_1 such that $\psi(a_1) = b_1$, and need to get some $\tilde{\beta} : \tilde{b_0} \Rightarrow a_1$ over β. This time we choose, in the input to the M-S condition, $a := a_1$, $b_1 := id$, $b_2 := b_0$, so $\beta : b_2 \Rightarrow \psi(a) \circ b_1$, as required, and can read off the lift accordingly. (Beware, you will get an extra lift, say x, of b_1 in your expression that you do not want, and cannot guarantee that it is the identity, however it is invertible, so you can adjust things to fit.)

Given that sketch, the other direction of the equivalence is easy. Assuming 1- and 2-arrow lifting, start with the M-S situation, lift b_1 using 1-arrow lifting, then $b_1 \circ \psi(a) = \psi(\tilde{b_1} \circ a)$, so we can apply 2-arrow lifting to β. ■

The advantage in having these two forms of the definition is that the M-S form is very neat from the categorical context, but the arrow lifting version is more easily seen to be the 2-groupoid version of the definition of fibration of crossed modules that we gave on page 209 and of the 'classical' epimorphism-condition for a 'fibration of simplicial groups'.

Moerdijk and Svensson, [100], also consider *cofibrations*. For the moment, we just need the corresponding condition for an object to be *cofibrant*.

Definitions: (i) A 2-group, $\mathcal{G}$, is *cofibrant in the Moerdijk-Svensson structure*, (we will say *M-S cofibrant*) if every trivial fibration $\mathcal{H} \to \mathcal{G}$, where $\mathcal{H}$ is a 2-groupoid, admits a section.

(ii) A crossed module, C, is *cofibrant* if the corresponding 2-group, $\mathcal{X}(\mathsf{C})$, is M-S cofibrant.

Proposition 57 *(Noohi, [106]) A crossed module* $\mathsf{C} = (C, P, \partial)$*, is cofibrant if and only if* P *is a free group.* ∎

The proof, which is given by Noohi, [106], is similar to that given above for Proposition 55. It can be safely **left to the reader**, except to note that it *does* require the use of the result that subgroups of free groups are free. (Analogues of this result in other categories than that of groups, would need reformulation to avoid the use of the analogous statement which may or may not be true in such settings.)

Example: For any crossed module, C, the pullback crossed module, $\varepsilon_P^*(\mathsf{C})$, or, equivalently $M(G\overline{W}(K(\mathsf{C})), 1)$, is cofibrant. We note also that it depends functorially on C and that there is a natural trivial fibration, $\varepsilon_P^*(\mathsf{C}) \to \mathsf{C}$.

Definition: (i) For C, a crossed module, a *cofibrant replacement for* C is cofibrant crossed module QC, together with a trivial fibration, $\mathsf{q} : \mathsf{QC} \to \mathsf{C}$.

(ii) A *cofibrant replacement functor (for crossed modules)* consists of a functor, $\mathsf{Q} : CMod \to CMod$, together with a natural transformation, $\mathsf{q} : \mathsf{Q} \to Id$, such that for each crossed module, C, $\mathsf{q}_\mathsf{C} : \mathsf{QC} \to \mathsf{C}$ is a cofibrant replacement for C.

The idea of cofibrant replacement given here is just the particular case for the context of crossed modules of a general notion from homotopical algebra. (We suggest that you look at a standard text on model categories and other ideas of homotopical algebra for further details. One such is Hovey's [76].) In a model category, as considered there, there are notions of weak equivalence, fibration and cofibration and thus of fibrant and cofibrant objects. For example, in the category of simplicial sets, considered with its usual model category structure, weak equivalences are what we would expect, that is, simplicial maps inducing isomorphisms of π_0 and all higher homotopy groups for all possible choices of base points. Fibrations are Kan fibrations and cofibrations are simplicial inclusions. All objects are cofibrant, but only the Kan complexes are fibrant. For simplicial groups, fibrations are the morphisms that are epimorphisms in each dimension, and the cofibrant objects are the simplicial groups that are free in each dimension.

For any model category, one can define cofibrant replacements as above, and, dually, fibrant replacements, and can prove that they always exist. They are the model categoric analogues of the projective and injective resolutions of more classical *homological* algebra and are similarly used to define *derived functors*. These, of course, are intimately related to cohomology theory, but we will not follow that link very far here, as our main use for this here is as an illustration and example of homotopy coherence.

For some of the theory of cofibrant replacements and total derived functors, look at the book by Hovey, [76], which is also an excellent introduction to the wider theory of model categories. (It is also useful to glance back at the original sources on homotopical algebra, in particular Quillen's orginal [113] and the related [114].)

If $\mathcal{C}$ is a model category and Q is a cofibrant replacement functor, the idea is that the value of the derived functor of some functor, $F : \mathcal{C} \to \mathcal{D}$, at an object, C, is obtained by looking at $F(QC)$ 'up to homotopy'. That *is* vague, but, in our context of weak maps, we have, for any given crossed module, D, a functor $CMod(-,\mathsf{D})$ from $CMod^{op}$ to $\ldots$, where? Actually 'to the category of groupoids' would be a suitable choice, as we have not only morphisms between crossed modules, but homotopies between them. There is also a groupoid of weak maps from C to D with weak natural transformations as the arrows. (This is **left to you** to look up in Noohi's papers, [106, 108] or to investigate yourselves.) As our functorial Q, given explicitly by $M(G\overline{W}(K(-),1))$, naturally gives weak maps, we come back to our question from earlier, which we can now ask with more exact terminology:

Suppose $\mathsf{q} : \mathsf{QC} \to \mathsf{C}$ *is a cofibrant replacement for* C*, and* $\psi : \mathsf{QC} \to \mathsf{D}$ *is a map of crossed modules, does* ψ *induce a weak map from* C *to* D*?*

We write $\mathsf{QC} = (QC_1, F, \partial_Q)$, and find that, as $\mathsf{q} : \mathsf{QC} \to \mathsf{C}$ is a trivial fibration, $QC_1 \cong F \times_P C = q_0^*(\mathsf{C})_1$. We thus have a lot of information about QC.

Next, apply the functorial construction to $\mathsf{q} : \mathsf{QC} \to \mathsf{C}$ to get

$$\begin{tikzcd}
\varepsilon_F^*(\mathsf{QC}) \arrow[r] \arrow[d] & \varepsilon_P^*(\mathsf{C}) \arrow[d] \\
\mathsf{QC} \arrow[r, "\mathsf{q}"'] & \mathsf{C}
\end{tikzcd}$$

as the two vertical morphisms and the bottom one are weak equivalences, so is the top. It is also a fibration. (In fact, it is the induced map which at level 1 is the obvious map,

$$FU(F) \times_F (F \times_P C) \to FU(P) \times_P C,$$

so is easily checked to be one.) It is thus a trivial fibration with cofibrant codomain. It is therefore split by some section,

$$\sigma : \varepsilon_P^*(\mathsf{C}) \to \varepsilon_F^*(\mathsf{QC}).$$

We can compose this with the natural morphism, $\mathsf{q}_{\mathsf{QC}} : \varepsilon_F^*(\mathsf{QC}) \to \mathsf{QC}$.

Now suppose $\psi : \mathsf{QC} \to \mathsf{D}$ is a morphism of crossed modules, then it gives a composite,

$$\varepsilon_P^*(\mathsf{C}) \xrightarrow{\sigma} \varepsilon_F^*(\mathsf{QC}) \xrightarrow{\mathsf{q}_{\mathsf{QC}}} \mathsf{QC} \xrightarrow{\psi} \mathsf{D}.$$

Clearly, there may be many sections of the map from $\varepsilon_F^*(\mathsf{QC})$ to $\varepsilon_P^*(\mathsf{C})$, so many different 'weak maps' would seem to correspond to a single $\psi : \mathsf{QC} \to \mathsf{D}$, but these weak maps only depend on ψ in the 'last composition'. If we look slightly more deeply, it becomes clear that they correspond to sections of $FU(F) \to FU(P)$, i.e., to choices of transversals for $FU(F) \to P$. This is known, 'standard', even 'classical' territory, and will be **left to you to explore**. The point is that two weak maps coming from different sections σ and σ' are likely to be 'homotopic' in some sense. (This is explored in the work of Noohi that we referred to earlier.) We summarise the above in the following:

Proposition 58 *If* $\mathsf{q} : \mathsf{QC} \to \mathsf{C}$ *is any cofibrant replacement for a crossed module, any crossed module morphism,* $\psi : \mathsf{QC} \to \mathsf{D}$*, induces a (usually non-unique) weak map of crossed modules from* C *to* D*.* ■

5.7.6 Weak maps: from cofibrant replacements to the algebraic form

It is not hard to start with a weak map, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, described as a pseudo-functor from $\mathcal{X}(\mathsf{C})$ to $\mathcal{X}(\mathsf{D})$, and to convert that description, via the nerves, to the algebraic description of f. (For instance, as the nerve of $\mathcal{X}(\mathsf{C})$ has P in one dimension and $C \times P \times P$ in the next, the values of f on these should give the f_0, f_1, and the pairing without too much bother.) **Leaving you to investigate** that later by yourself, let us pass further into the simplicial description and use the functorial cosimplicial replacement, $\varepsilon^*_P(\mathsf{C})$, so that we specify f by a crossed module morphism,

$$\mathsf{f} : \varepsilon^*_P(\mathsf{C}) \to \mathsf{D}.$$

(We will write $\mathsf{D} = (D_1, D_0\partial_{\mathsf{D}})$.) This gives us a square

$$\begin{tikzcd} F \times_P C \arrow[r, "f_1"] \arrow[d, "\partial"'] & D_1 \arrow[d, "\partial_{\mathsf{D}}"] \\ F \arrow[r, "f_0"'] & D_0 \end{tikzcd}$$

where we have written F for $FU(P)$. The elements of $F \times_P C$ are pairs (ω, c), where $\varepsilon_P(\omega) = \partial_{\mathsf{C}} c$, thus ω is a word in generators corresponding to elements of P. We will write (p) for the generator coming from $p \in P$.

Surprisingly enough the f_0 in this corresponds almost exactly to the f_0 in the usual algebraic description. There is a small difference, $f_0(p)$ in the latter description is $f_0((p))$ in the former one, so is the composite of the cofibrant replacement's f_0 with the set theoretic section, η_P, of the epimorphism, $\varepsilon_P : F \to P$, given by 'p goes to (p)', in other words, with the unit of the free-forget adjunction.

Notationally we need to distinguish the two, so will write f^{cr}_i for the different levels of the crossed module morphism, $f : \varepsilon^*_P(\mathsf{C}) \to \mathsf{D}$, the superfix 'cr' standing for 'cofibrant replacement', of course. This notation will be a temporary one. We thus have

$$f_0(p) = f^{cr}_0((p)).$$

We need to obtain $\langle -, - \rangle : P \times P \to D_1$, and $f_1 : C \to D_1$ and these must satisfy certain rules; see the definition on page 201. The basic ones are $\partial_{\mathsf{D}} f_1 = f_0 \partial_{\mathsf{C}}$, and the two 'almost a homomorphism' conditions. The one for f_0 gives

$$f_0(p_2p_1) = \partial\langle p_2, p_1\rangle^{-1} f_0(p_2) f_0(p_1).$$

This gives us a lever to get at $\langle p_2, p_1\rangle$. For any pair of elements, p_2, p_1 in P, we have a cocycle

$$(p_2)(p_1)(p_2p_1)^{-1} \in FU(P) = F$$

and this is in the kernel of ε_P. As a result, there is an element

$$\{p_2, p_1\} = ((p_2)(p_1)(p_2p_1)^{-1}, 1) \in F \times_P C.$$

We look at

$$f_1^{cr}\{p_2, p_1\} \in D_1.$$

We have

$$\partial_{\mathsf{D}} f_1^{cr}\{p_2, p_1\} = f_0^{cr}\partial\{p_2, p_1\} = f_0(p_2)f_0(p_1)f_0(p_2p_1)^{-1},$$

so if we take $\langle p_2, p_1\rangle := f_1^{cr}\{p_2, p_1\}$, we get the 'almost a homomorphism' condition for f_0.

What about that for f_1? Well, we have yet to write down some f_1 in terms, perhaps, of f_1^{cr}, but if we have $c \in C$, then we clearly have an element $((\partial_{\mathsf{C}} c), c) \in F \times_P C$, so it is a fairly safe bet that $f_1(c)$ will be $f_1^{cr}((\partial_{\mathsf{C}} c), c)$, (or possibly its inverse, since directions can easily get reversed with the different conventions, and it does not pay to be too sure in advance of detailed checking!) The obvious thing to do is to try it in the W2 'almost a homomorphism' condition for f_1, again see the discussion around page 201. In fact, we note

$$\begin{aligned} ((\partial_{\mathsf{C}} c_2), c_2).((\partial_{\mathsf{C}} c_1), c_1) &= ((\partial_{\mathsf{C}} c_2)(\partial_{\mathsf{C}} c_1), c_2c_1) \\ &= ((\partial_{\mathsf{C}} c_2)(\partial_{\mathsf{C}} c_1)(\partial_{\mathsf{C}}(c_2c_1))^{-1}, 1)((\partial_{\mathsf{C}}(c_2c_1), c_2c_1), \end{aligned}$$

so, mapping this via f_1^{cr} gives

$$f_1(c_2)f_1(c_1) = \langle \partial c_2, \partial c_1\rangle.f_1(c_2c_1),$$

as required.

Of course, we will need to check the other two conditions, but that is **left to you**. (The cocycle condition is easy to check, the whiskering conditions do require some work. You might start by checking what the action of F on $F \times_P C$ is.) We have proved (modulo your checking):

Proposition 59 *Given a morphism $f^{cr} : \varepsilon_P^*(\mathsf{C}) \to \mathsf{D}$, the structure*

- $f_0 : P \to D_0$ *given by* $f_0(p) = f_0^{cr}((p))$;
- $f_1 : C \to D_1$ *given by* $f_1^{cr}(\partial_{\mathsf{C}} c), c)$;
- $\langle -, -\rangle : P \times P \to D_1$ *given by* $\langle p_2, p_1\rangle := f_1^{cr}\{p_2, p_1\}$, *where* $\{p_2, p_1\} = ((p_2)(p_1)(p_2p_1)^{-1}, 1)$,

specifies a weak map, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, *(in the algebraic description format).* ∎

5.7.7 Butterflies

We have, when discussing the algebraic definition of a weak map, pointed out the similarities of certain structure with the cocycle description of group extensions and, thus, of group cohomology. For instance, f_0 and $\langle -, -\rangle$ together yield something very like a weak action of P (on D). The cocycle condition, also, is very reminiscent of the conditions on the factor set, $f : G \times G \to K$, that ensure associativity of the multiplication if reconstructing the middle term of the extension from the two ends, together with the weak action and the factor set. This suggests that there should be an extension associated with a weak map.

Collecting up evidence, we have our 'factor set'-like pairing, $\langle -, - \rangle$, going, in our typical situation, from $P \times P$ to D_1. This would correspond to a group extension

$$D_1 \stackrel{\iota}{\to} E \stackrel{\rho}{\to} P,$$

and the cocycle condition suggests that we use $f_0 : P \to D_1$ to get a weak action of P on D_1, that is, looking at the cocycle condition and comparing it with the factor set condition (page 44), we need to get P to 'act' on D_1, and we can use f_0 to get from P to D_0 and then use the action of D_0 on D_1 to get something that might work. In other words, we will interpret ${}^{f_0(p)}x$ for $p \in P$ and $x \in D_1$ as the analogue of the weak action in the extension.

To construct the middle term, E, (as in section 2.3.1), we take the *set* $D_1 \times P$ and give it a multiplication

$$(x_1, p_1)(x_2, p_2) = (x_1.{}^{f_0(p_1)}x_2.\langle p_1, p_2\rangle, p_1p_2).$$

The checking that this is associative, etc., is quite easy, but we will give it in some detail as it is neat and shows how the properties of the pseudo-functor defining the weak map are transformed into quite usual properties of the object, E. This checking is, of course, quite standard in the theory of group extensions.

Lemma 41 *The above multiplication is associative.*

Proof: We calculate

$$\begin{aligned}(x_1, p_1)((x_2, p_2)(x_3, p_3)) &= (x_1, p_1)(x_2.{}^{f_0(p_2)}x_3\langle p_2, p_3\rangle, p_2p_3)\\ &= (x_1.{}^{f_0(p_1)}x_2.{}^{f_0(p_1)f_0(p_2)}x_3.{}^{f_0(p_1)}\langle p_2, p_3\rangle\langle p_1, p_2p_3\rangle, p_1p_2p_3).\end{aligned}$$

(It is worth noting that terms that exist in the cocycle condition for $\langle -, - \rangle$ are occurring naturally here.) The 'other side' gives

$$\begin{aligned}((x_1, p_1)(x_2, p_2))(x_3, p_3)) &= (x_1.{}^{f_0(p_1)}x_2\langle p_1, p_2\rangle, p_1p_2)(x_3, p_3)\\ &= (x_1.{}^{f_0(p_1)}.\langle p_1, p_2\rangle.{}^{f_0(p_1p_2)}x_3\langle p_1p_2, p_3\rangle, p_1p_2p_3).\end{aligned}$$

Comparing the two expressions, we can match up corresponding parts leaving, in the first expression,

$${}^{f_0(p_1)f_0(p_2)}x_3.{}^{f_0(p_1)}\langle p_2, p_3\rangle\langle p_1, p_2p_3\rangle,$$

which rewrites, using 'cocycle', to

$${}^{f_0(p_1)f_0(p_2)}x_3.\langle p_1, p_2\rangle\langle p_1p_2, p_3\rangle.$$

The last term matches with one in the equivalent position in the second expression. We then attack $f_0(p_1)f_0(p_2)$, using 'almost a homomorphism', giving $\partial\langle p_1, p_2\rangle f_0(p_1, p_2)$. We finally use the Peiffer identity, so

$$\begin{aligned}{}^{f_0(p_1)f_0(p_2)}x_3.\langle p_1, p_2 &= {}^{\partial\langle p_1,p_2\rangle f_0(p_1,p_2)}x_3.\langle p_1, p_2\rangle\\ &= \langle p_1, p_2\rangle.{}^{f_0(p_1p_2)}x_3.\langle p_1, p_2\rangle^{-1}.\langle p_1, p_2\rangle\\ &= \langle p_1, p_2\rangle.{}^{f_0(p_1p_2)}x_3,\end{aligned}$$

as hoped. ∎

The identity for the multiplication is clearly $(1,1)$, so we certainly have a monoid. What about inverses? We are given (x,p), and so need to solve

$$(y,q).(x,p) = 1.$$

This gives $q = p^{-1}$ and

$$y = \langle p^{-1}, p\rangle^{f_0(p^{-1})} x^{-1}$$

and so

$$(x,p)^{-1} = (\langle p^{-1}, p\rangle^{f_0(p^{-1})} x^{-1}, p^{-1}).$$

Remark: Of course, we know by standard elementary arguments that this 'left inverse' is also a 'right inverse', but **it is quite interesting to calculate the product**, showing

$$(x,p)(\langle p^{-1}, p\rangle^{f_0(p^{-1})} x^{-1}, p^{-1}) = (1,1)$$

directly. 'Interesting'? Yes, because it presents some useful calculations that otherwise would not come to the surface this early in an investigation. For instance, we have both $\langle p^{-1}, p\rangle$ and $\langle p, p^{-1}\rangle$, occurring in the formulae. What is their relationship?

Lemma 42

$$\langle p, p^{-1}\rangle = {}^{f_0(p)}\langle p^{-1}, p\rangle.$$

■

The proof follows from the cocycle condition using $p_1 = p_3 = p$ and $p_2 = p^{-1}$.

Another such result is

Lemma 43

$$f_0(p) f_0(p^{-1}) = \partial\langle p, p^{-1}\rangle.$$

■

This is, of course, an immediate consequence of 'almost a homomorphism' and 'normalization', but, for calculations, is very useful to have explicitly stated.

We have now verified that E is a group - which was obvious from the classical theory of factor sets and has nothing specific to do with weak maps or crossed modules. We record the structural maps for convenience:

in $D_1 \xrightarrow{\iota} E \xrightarrow{\rho} P$*, the maps are given by* $\iota(x) = (x,1)$*,* $\rho(x,p) = p$*.*

These are easily seen to be homomorphisms.

All that is standard Schreier theory of factor sets and extensions and gives us a diagram, (a 'partial butterfly'),

$$\begin{tikzcd}
C \arrow[dd, "\partial"'] & & D_1 \arrow[dl, "\iota"'] \arrow[dd, "\partial_D"] \\
& E \arrow[dl, "\rho"'] & \\
P & & D_0
\end{tikzcd}$$

In Noohi's theory of papillons (butterflies), (cf. [106] and [2]), we have the following definition:

Definition: Let $\mathsf{C} = (C, P, \partial)$ and $\mathsf{C}' = (C', P', \partial')$ be two crossed modules. By a *papillon*, or *butterfly*, from C to C', we mean a commutative diagram of groups

$$\begin{tikzcd} C \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & C' \arrow[dl, "\iota"'] \arrow[dd, "\partial_{\mathsf{C}'}"] \\ & E \arrow[dl, "\rho"] \arrow[dr, "\lambda"'] & \\ P & & P' \end{tikzcd}$$

in which the diagonals are complexes of groups (so $\lambda\kappa$ and $\rho\iota$ are trivial homomorphisms), the NE-SW sequence,

$$C' \xrightarrow{\iota} E \xrightarrow{\rho} P,$$

is short exact (hence is a group extension), $Ker\,\rho = Im\,\iota$, and, moreover, for all $e \in E, c \in C$ and $c' \in C'$, we have

$$\iota({}^{\lambda(e)}c') = e\iota(c')e^{-1},$$

and

$$\kappa({}^{\rho(e)}c) = e\kappa(c')e^{-1}.$$

As 'papillons' are introduced, in [106] and [2], as a way to handle weak maps, we should be able to complete our partial butterfly to a full one by defining a NW-SE complex. The first map, $\kappa : C \to E$, must be something like $\kappa(c) = (f_1(c), \partial c)$, as the usual rule in these situations is 'build it simply from the parts that you have'. That, however, does not quite work. (This may be due to a question of conventions when representing elements of E in the form (x, p), and some different choice might result in the 'fault' disappearing, however I doubt it, but have no evidence 'one way or t'other', - **it is left as a challenge to the reader to shed some light on this!**) Surprisingly enough, what happens with that attempt gives us the clue to resolving the problem.

(To simplify notation slightly, we will usually write ∂ for the boundary in all the crossed modules involved. Context in each case diminishes the risk of confusion.)

Define $\kappa(c) = (f_1(c)^{-1}, \partial c)$.

Proposition 60 *Defined by this, $\kappa : C \to E$ is a homomorphism satisfying*

$$\kappa({}^{\rho(e)}c) = e\kappa(c)e^{-1}.$$

Proof: (This is another of the calculatory verification proofs that could be very safely left to the reader - but, because of strange inversion in the first factor of κ, it is interesting to see how this works.)

We take $c_1, c_2 \in C$,

$$\begin{aligned} \kappa(c_2c_1) &= (f_1(c_2c_1)^{-1}, \partial c_2c_1) \\ &= ((\langle \partial c_2, \partial c_1\rangle^{-1} f_1(c_2)f_1(c_1))^{-1}, \partial c_2\partial c_1) \\ &= (f_1(c_1)^{-1}f_1(c_2)^{-1}\langle \partial c_2, \partial c_1\rangle, \partial c_2\partial c_1), \end{aligned}$$

whilst

$$\begin{aligned}\kappa(c_2)\kappa(c_1) &= (f_1(c_2)^{-1}, \partial c_2)(f_1(c_1)^{-1}, \partial c_1)\\ &= (f_1(c_2)^{-1}.{}^{f_0(\partial c_2)}f_1(c_1)^{-1}\langle \partial c_2, \partial c_1\rangle, \partial c_2 \partial c_1).\end{aligned}$$

Using that $f_0\partial = \partial f_1$, and the Peiffer identity completes the proof that these are equal.

To prove the second condition, it helps to note the following lemma.

Lemma 44 *For any $c \in C$, $c' \in C'$, $[\iota(c'), \kappa(c)] = 1$.*

Proof: We note $\iota(c') = (c', 1)$, whilst $\kappa(c) = (f_1(c)^{-1}, \partial c)$. Now

$$(c', 1)(f_1(c)^{-1}, \partial c) = (c' f_1(c)^{-1}, \partial c),$$

since $\langle 1, \partial c\rangle = 1$ and $f_0(1) = 1$. On the other hand,

$$(f_1(c)^{-1}, \partial c)(c', 1) = (f_1(c)^{-1}.{}^{f_0(\partial c)}c', \partial c),$$

but, as we have used so many times $f_0\partial = \partial f_1$, so the Peiffer identity gives ${}^{f_0(\partial c)}c' = f_1(c)c' f_1(c)^{-1}$ and the lemma follows. ■

Because of this and the fact that any $(x, p) \in E$ can be decomposed as $(x, 1)(1, p)$, it suffices to prove the result for $e = (1, p)$. This is quite easy and goes as follows:

We first work out $\kappa({}^pc)$. This is $(f_1({}^pc)^{-1}, p.\partial c.p^{-1})$, so we first need $f_1({}^pc)$, but the formula from earlier gave

$$f_1({}^pc) = \langle p.\partial c, p^{-1}\rangle^{-1}\langle p, \partial c\rangle^{-1} f_1(c)\langle p, p^{-1}\rangle,$$

so our 'target formula' should be

$$\kappa({}^pc) = (f_1({}^pc)^{-1}, p\partial c p^{-1}) = (\langle p, p^{-1}\rangle^{-1} f_1(c)^{-1}\langle p, \partial c\rangle\langle p.\partial c, p^{-1}\rangle, p.\partial c.p^{-1}).$$

We thus have to show that this is the result of conjugating $\kappa(c)$ by $(1, p)$. Now

$$\begin{aligned}(1,p)(f_1(c)^{-1}, \partial c)(1,p)^{-1} &= (1,p)(f_1(c)^{-1}, \partial c)(\langle p^{-1}, p\rangle^{-1}, p^{-1})\\ &= (1,p)(f_1(c)^{-1}.{}^{f_0(\partial c)}\langle p^{-1}, p\rangle^{-1}\langle \partial c, p^{-1}\rangle, \partial c.p^{-1})\\ &= (1,p)(f_1(c)^{-1}.{}^{\partial f_1(c)}\langle p^{-1}, p\rangle^{-1}\langle \partial c, p^{-1}\rangle, \partial c.p^{-1})\\ &= (1,p)(f_1(c)^{-1}.f_1(c)\langle p^{-1}, p\rangle^{-1} f_1(c)^{-1}\langle \partial c, p^{-1}\rangle, \partial c.p^{-1}) \quad \text{by Peiffer}\\ &= (1,p)(\langle p^{-1}, p\rangle^{-1} f_1(c)^{-1}\langle \partial c, p^{-1}\rangle, \partial c.p^{-1})\\ &= ({}^{f_0(p)}\langle p^{-1}, p\rangle^{-1}.{}^{f_0(p)}f_1(c)^{-1}.{}^{f_0(p)}\langle \partial c, p^{-1}\rangle\langle p, \partial c.p^{-1}\rangle, p.\partial c.p^{-1}),\end{aligned}$$

but ${}^{f_0(p)}\langle p^{-1}, p\rangle^{-1} = \langle p, p^{-1}\rangle^{-1}$, as we saw earlier, and the cocycle rule tells us that

$$\langle p, \partial c\rangle\langle p.\partial c, p^{-1}\rangle = {}^{f_0(p)}\langle \partial c, p^{-1}\rangle\langle p, \partial c.p^{-1}\rangle,$$

so the verification is complete. ■

We next need $\lambda : E \to P'$. If $e = (x, p) \in E$, both x and p map easily into P' and, as there is nothing to choose between them, ..., we use them both and try $\lambda(x, p) = \partial x.f_0(p)$.

Lemma 45 *Thus defined, $\lambda : E \to P'$ is a homomorphism, and $\lambda\kappa$ is the trivial homomorphism, (so NW-SE is a group complex).*

Proof: Left to you. ■

We must also check the validity of ι's credentials!

Proposition 61 *Defining $\iota : C' \to E$ by $\iota(x) = (x, 1)$, ι is a homomorphism, satisfying: for all $e \in E$, and $c' \in C'$*

$$\iota({}^{\lambda(e)}c') = e\iota(c')e^{-1},$$

Proof: The first part is easy, since $\iota(x_2x_1) = (x_2x_1, 1)$, whilst the multiplication fromula in E gives the same thing for $\iota(x_2)\iota(x_1)$.

We next note that, if $e = (x, p)$, then $\lambda(e) = \partial x.f_0(p)$, so

$$\iota({}^{\lambda(e)}c') = ({}^{\partial x.f_0(p)}c', 1) = (x.{}^{f_0(p)}c'.x^{-1}, 1),$$

whilst

$$\begin{aligned}(x, p)(c', 1)(x, p)^{-1} &= (x.{}^{f_0(p)}c', p)(\langle p^{-1}, p\rangle^{-1}.f_0(p^{-1})x^{-1}, p^{-1}) \\ &= (x.{}^{f_0(p)}c'.{}^{f_0(p)}\langle p^{-1}, p\rangle^{-1}.f_0(p)f_0(p^{-1})x^{-1}\langle p, p^{-1}\rangle, 1).\end{aligned}$$

We have $f_0(p)f_0(p^{-1}) = \partial\langle p, p^{-1}\rangle^{-1}$, so this simplifies to

$$(x.{}^{f_0(p)}c'.{}^{f_0(p)}\langle p^{-1}, p\rangle^{-1}\langle p, p^{-1}\rangle x^{-1}\langle p, p^{-1}\rangle^{-1}\langle p, p^{-1}\rangle, 1),$$

and using that ${}^{f_0(p)}\langle p^{-1}, p\rangle = \langle p, p^{-1}\rangle$ gives the result. ■

We summarise:

Proposition 62 *From a weak map, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, the above construction gives a papillon, $\mathbb{f}$,*

$$\begin{tikzcd} C \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & C' \arrow[dl, "\iota"'] \arrow[dd, "\partial_{\mathsf{C}'}"] \\ & E \arrow[dl, "\rho"] \arrow[dr, "\lambda"'] & \\ P & & P' \end{tikzcd}$$

from C to C'. ■

What about a converse to this? Does a papillon yield a weak map in some nice way? Recalling that the NE-SW sequence is a group extension, if we pick a section for ρ and compose it with λ, we should get a possible $f_0 : P \to P'$, and a 'factor set' pairing $\langle -, \rangle : P \times P \to C'$. We will also obtain a decomposition of E as a product of P and C' at the underlying set level, and hence can use κ and the set theoretic projection to C' to obtain a suitable f_1. we will leave the investigation of this as **an extended exercise for you.**

Of course, different sections of ρ may yield different f_0s, so we need a notion of morphisms of papillons and there is an obvious candidate.

Definition: If C, and C' are two crossed modules and $\mathbb{f}$ and $\mathbb{f}'$ are two papillons from C to C' (with central group E' in $\mathbb{f}'$, and with 'primes' on the morphisms, κ', etc.), then a morphism from $\mathbb{f}$ to $\mathbb{f}'$ is a homomorphism, $\varphi : E \to E'$, such that $\kappa' = \varphi\kappa$, etc., thus making the evident diagram commute.

Such diagrams compose in the obvious way. This gives a category, in fact, a groupoid because of the following:

Lemma 46 *Any morphism $\varphi : \mathbb{f} \to \mathbb{f}'$ between two papillons, $\mathbb{f}$ to $\mathbb{f}'$, as above, is an isomorphism.*

Proof: This is clear from the fact that φ yields a map of extensions

$$\begin{tikzcd}
1 \arrow[r] & C' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi"] & P \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & C' \arrow[r] & E' \arrow[r] & P \arrow[r] & 1
\end{tikzcd}$$

and any such φ must be an isomorphism by the usual 5-lemma argument on short exact sequences. (**Really you should check that the inverse of φ (as a group homomorphism) gives a *morphism of papillons* inverse to φ itself,** but that is more or less obvious.) ■

The category of papillons from C to C' is thus a groupoid, but so is the category of weak maps and 'weak natural transformations' between them. It may be useful to **investigate the relationships between them**. This is one of the themes of Noohi's work, [108]. His joint work with Aldrovandi, [2], further explores this in the context of stacks (of groupoids) and so is also highly relevant to our overall themes.

5.7.8 . . . and the strict morphisms in all that?

As we noted much earlier, any morphism of crossed modules gives a 2-functor of the corresponding 2-groups, that is, a strict, rather than an op-lax, '2-functor'. It would be very bizarre if the fact that a given 'weak morphism' was actually a 'strict' one was not evident in the descriptions. That is not to claim that we should be necessarily able to glance at some weak map and decide quickly if it is actually a strict one. No, we should perhaps expect to have to do a little work, to test 'things' somewhat. What 'things' however?

We start with the description via nerves. Any strict $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ induces a simplicial map,

$$Ner(\mathsf{f}) : Ner(\mathsf{C}) \to Ner(\mathsf{D}),$$

both for $Ner(\mathsf{C})$ interpreted as $Ner_{h.c.}(\mathcal{X}(\mathsf{C}))$ and as $\overline{W}(K(\mathsf{C}))$. Does $Ner(\mathsf{f})$ have any identifiable property over arbitrary simplicial maps between two nerves (and thus over weak maps)?

The secret identifier is 'preservation of thinness'. We have had several definitions of the nerve of a crossed module. We had $\overline{W}(K(\mathsf{C}))$, $Ner_{h.c.}(\mathcal{X}(\mathsf{C}))$, but also $Crs(\pi(-), \mathsf{C})$, that is, the simplicial set of crossed complex maps from the various $\pi(n)$ to C, where this $\pi(n)$ is the free crossed complex on the n-simplex, $\Delta[n]$, as was briefly discussed on page **??**. That '*singular complex*' version is very useful, and we have not yet exhausted its possibilities, far from it, but neither have we really done it justice, yet!

These various nerves are isomorphic, and so are all T-complexes. The thin elements in the last description are those $\tau : \pi(n) \to \mathsf{C}$, which map the generator corresponding to ι_n, the top level non-degenerate n-simplex of $\Delta[n]$, to an identity element. The elements of each $Ner(\mathsf{C})_n$ for $n > 2$ are all thin since, as a crossed complex, C is trivial in dimensions greater than 2. (Beware of indexing conventions! Yes, we do need 2, here not 1.)

If we use the h. c. / geometric nerve form, a general 2-simplex, τ in $Ner(\mathsf{C})$ has form,

$$\tau = (x_0, x_1, x_2; x(012) : x_1 \Rightarrow x_0x_2),$$

where, thus, $x(012) = (c, x_1)$ with $\partial c.x_1 = x_0x_2$. The interpretation of the condition that $\tau(\iota_2)$ be the identity is that c is the identity of C, i.e., the 2-simplex is 'really' in $Ner(P)$, in other words, it commutes, $x_1 = x_0x_2$.

The thin 1-simplices will be the degenerate ones. What about thin 3-simplices? We know $Ner(\mathsf{C})$ is 3-coskeletal, and this came out to be because there were no non-identity 3-cells in the 2-groupoid, $\mathcal{X}(\mathsf{C})$, and, yes, that means that any $\tau : \pi(3) \to \mathsf{C}$ must send the generator corresponding to ι_3 to the identity element, 'there ain't nothing else there to map it to!'. We thus have all 3-simplices are thin, as are all higher dimensional simplices.

Remark: It is a good **exercise** to **define** thinness for these simplices in this way (i.e., without explicit reference to crossed complexes or to $\pi(n)$), and then to check directly that the result is a T-complex (definition and discussion starting on page 30 if you need it). Another **useful exercise** is to write down what $\pi(n)$ is **in 'gory' detail** and to explore the isomorphisms that we mentioned above between the descriptions of $Ner(\mathsf{C})$ given here and the crossed complex based one as a 'singular complex'.

To continue this exploration of 'strictness' of morphisms, we probably need a definition:

Definition: A simplicial map, $f : Ner(\mathsf{C}) \to Ner(\mathsf{D})$, between the geometric nerves of two crossed modules, *preserves thin elements* or, more simply, *preserves thinness* if, for each n, and each thin n-simplex, $t \in Ner(\mathsf{C})_n$, $f_n(t)$ is thin in $Ner(\mathsf{D})$.

Remark: We should comment that preservation of thinness really devolves down to checking that a map preserves thin 2-simplices. The thin 1-simplices are just the degenerate ones, so they will be preserved by any simplicial map, whilst, above dimension 2, all simplices are thin, so preservation is automatic!

We showed (Proposition 178) how a simplicial map, $f : Ner(\mathsf{C}) \to Ner(\mathsf{D})$, induced the data for a pseudo-functor,

$$\mathcal{F} = (F, \varphi) : \mathcal{X}(\mathsf{C}) \to \mathcal{X}(\mathsf{D}).$$

(We will not need to use the detailed notation from there for the limited discussion that we will give here, so will abuse notation enormously!) Translating that data, in the algebraic / combinatorial format, we look at $(p_0, p_0p_2, p_2; id) \in Ner(\mathsf{C})$ and obtain

$$f_2(p_0, p_0p_2, p_2; id) = (f_0(p_0), f_0(p_0p_2), f_0(p_2); (\langle p_2, p_0\rangle, f_0(p_0p_2)))$$

with $\partial\langle p_2, p_0\rangle f_0(p_0p_2) = f_0(p_0)f_0(p_2)$.

If f preserves thinness, then $\langle p_2, p_0\rangle$ is trivial, i.e., the identity in D, so f_0 is a homomorphism, as is f_1, and, by the post-whiskering axiom, $f_1({}^pc) = {}^{f_0(p)}f_1(c)$, so f is a (strict) morphism of crossed modules, as required. ■

Clearly, if $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ is a crossed module morphism, then it preserves thinness (in all dimensions). (Just **check it**.)

This raises an interesting **question**. Is there a simple example of a weak (and not strict) morphism of crossed modules, having both f_0 and f_1 group homomorphisms? In such a case, all the $\partial\langle p_1, p_2\rangle$ and $\langle\partial c_1, \partial c_2\rangle$ would be trivial, but would it be possible to have some $\langle p_1, p_2\rangle$ non-trivial? The obvious place to look first would be with modules thought of as crossed modules, so the various ∂ would be trivial.

The above more or less indicates what a strict morphism has that a weak one does not, from the point of view of nerves. What about defining weak maps via cofibrant replacements? If we start with a strict morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, and a cofibrant replacement, $\mathsf{q} : \mathsf{Q} \to \mathsf{C}$, then there is clearly a morphism,

$$\mathsf{fq} : \mathsf{Q} \to \mathsf{D},$$

which will be a weak map from C to D, or, more exactly, will be one if Q is the natural functorial cofibrant replacement, and, more generally, will *give* a weak map, determined up to equivalence. Conversely, given some $\mathsf{g} : \mathsf{Q} \to \mathsf{D}$, it will correspond to a strict map if g factors through q giving a 'complementary' morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, ... Uniqueness, etc, of the factorisation is **left to you** to analyse.

Finally, what sort of papillon / butterfly corresponds to a strict morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$? We know that f corresponds to a pairing, $\langle -, -\rangle : P \times P \to C'$, which, here, is trivial. It follows that the NE-SW extension of the papillon will be split, with, as a result, $E \cong C' \rtimes P$, since $\langle -, -\rangle$ was a factor set for it.

This gives a papillon:

$$\begin{tikzcd}
C \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & C' \arrow[dl, "\iota"'] \arrow[dd, "\partial_{\mathsf{C}'}"] \\
& C' \rtimes P \arrow[dl, "\rho"] \arrow[dr, "\lambda"'] & \\
P & & P'
\end{tikzcd}$$

in which ρ is a split epimorphism.

Now we can go back. First the obvious definition:

Definition: A papillon, as above, in which the NE-SW extension is split (with given splitting) will be called a *split papillon.*

Suppose we have such a split papillon, with $s : P \to C' \rtimes P$, the chosen splitting. (Of course, as soon as we choose a splitting, we are choosing an isomorphism of the central object, E, of the papillon and a semidirect product representation of it. Consequently, if we write $C' \rtimes P$ for the centre term of a papillon, we are not only identifying that group, but are specifying the splitting (namely $s(p) = (1, p)$) and a host of other information. This does lead to a certain redundancy of notation and, perhaps, of terminology, but, hopefully, is clearer in terms of the exposition.) The decomposition as $C' \rtimes P$ also gives us a set theoretic projection from $C' \rtimes P$ to C', which we will denote by d. (This satisfies

$$d((c'_1, p_1)(c'_2, p_2)) = c'_1.{}^{p_1}c'_2,$$

whilst, of course, $d(c'_1, p_1)d(c'_2, p_2) = c'_1.c'_2$, so d is not a homomorphism. It is a derivation.) We want to construct a morphism of crossed modules,

$$\mathsf{f} : \mathsf{C} \to \mathsf{C}'.$$

There is an obvious $f_0 : P \to P'$, given by λs, but what about an $f_1 : C \to C'$?

There seem to be only a few possibilities handed to us if we are to use just the 'building blocks' provided. We know that the left 'wing' of the papillon commutes, so $\kappa(c) = (k(c), \partial c)$ and perhaps this mapping, $k : C \to C'$, is what we need.

Before we go further, however, we should look back at how we went *from* weak maps *to* 'papillons'. We took $\kappa(c) = (f_1(c)^{-1}, \partial c)$, so *that* suggests that $k(c)$ is not exactly what we want, rather $k(c)^{-1}$ should be the thing we look at.

(If we look at the fact that κ itself *is* a homomorphism, then k satisfies a derivation type formula,

$$k(c_2c_1) = k(c_2).{}^{\partial c_2}k(c_1),$$

rather than being a homomorphism. We are in the context of crossed modules, so action by a boundary element, such as ∂c_2, easily converts to conjugation, but the above seems to then end up with the wrong order for things to cancel as we might hope. This again suggests that the idea of the 'inverse of k' is a good one to follow up.)

Given this, we will bravely set $f_1(c) := k(c)^{-1}$ and charge into the attack! First, however, let us make a cunning observation. The above choice looks good, as we said, since then

$$\kappa(c) = (f_1(c)^{-1}, \partial c)$$

as before, so

$$\kappa(c) = (f_1(c)^{-1}, 1)(1, \partial c) = \iota(f_1(c))^{-1}.s(\partial c).$$

Rearranging this gives

$$\iota(f_1(c)) = s(\partial c)\kappa(c)^{-1},$$

we further note that (i) ι is a monomorphism, and (ii), and, in all generality, $[\iota(c'), \kappa(c)] = 1$, since $\rho\iota(c') = 1$ implies that

$$\iota(c')\kappa(c)\iota(c')^{-1} = \kappa({}^{\rho\iota(c')}c) = \kappa(c).$$

(In case you are wondering, it should be noted, that we had previously checked this only for a papillon coming from a weak map, so we *did* need to check it independently!)

Proposition 63 *Given a split papillon, as above, defining $f_0 = \lambda s$ and f_1 given by $\iota f_1(c) = s(\partial c)\kappa(c)^{-1}$, then (f_1, f_0) gives a morphism,* $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$.

Proof: We have to check three things:

(a) $\partial f_1 = f_0\partial$;

(b) f_1 is a homomorphism (as we have already checked that f_0 is one);

(c) for all $c \in C$ and $p \in P$,

$$f_1({}^p c) = {}^{f_0(p)} f_1(c).$$

Starting with (a), we have

$$\partial f_1(c) = \lambda\iota f_1(c) = \lambda s(\partial c),$$

since $\lambda\kappa$ is trivial, hence $\partial f_1(c) = f_0\partial(c)$.

Now (b), let c_1, $c_2 \in C$,

$$\begin{aligned} \iota f_1(c_2c_1) &= s\partial(c_2c_1).\kappa(c_2c_1)^{-1} \\ &= s\partial(c_2)s\partial(c_1)\kappa(c_1)^{-1}\kappa(c_2)^{-1}. \end{aligned}$$

(We know what we want this to be, so force it into the right shape with a rewrite.) It equals

$$s\partial(c_2)\kappa(c_2)^{-1}(\kappa(c_2)s\partial(c_1).\kappa(c_1)^{-1}\kappa(c_2)^{-1}) = s\partial(c_2)\kappa(c_2)^{-1}.\kappa(c_2).\iota f_1(c_1).\kappa(c_2)^{-1},$$

but κ and ι "commute", as we saw, so this is $\iota f_1(c_2)\iota f_1(c_1)$, as hoped for.

Finally (c), we take $p \in P$, $c \in C$

$$\begin{aligned} \iota f_1({}^p c) &= s\partial({}^p c).\kappa({}^p c)^{-1} \\ &= s(p\partial c.p^{-1}).\kappa({}^{\rho s(p)}c)^{-1}, \end{aligned}$$

since $p = \rho s(p)$. We use the condition on κ relative to the action of the $\rho(e)$s to get that this is

$$\begin{aligned} s(ps(\partial c)s(p)^{-1}.(s(p)\kappa(c)^{-1}s(p)^{-1}) &= s(p)(s(\partial c).\kappa(c)^{-1})s(p)^{-1} \\ &= s(p)\iota f_1(c)s(p)^{-1}. \end{aligned}$$

We now invoke the condition on ι relative to the action of the $\lambda(e)$s. This becomes $\iota({}^{\lambda s(p)} f_1(c))$, i.e., $\iota({}^{f_0(p)} f_1(c))$. Using that ι is a monomorphism, we get

$$f_1({}^p c) = {}^{f_0(p)} f_1(c),$$

as required. ■

We thus have strict morphisms correspond to split papillons. To be complete in this, we must note that a split papillon may have different splittings, so does a split papillon correspond to several *different* weak morphisms? Clearly, if it does, then these should be equivalent / homotopic. This is **left to you** to check up on and to investigate further. The papers, [106, 108] and [2], will give some ideas about what to expect, but do not expect them to provide all the answers!

It should also be clear that a weak equivalence of crossed modules should correspond to a papillon in which the NW-SE sequence is also exact. Noohi's discussion in [108] goes into this, and this is **suggested as another investigation**. His treatment does not take quite the same route through the ideas as we have, so there are quite a few details to supply … **over to you**.

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