Tutorial on
PROOF NETS
(Logic and Interaction 2012, week 2)

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LL sequent calculus

$$\frac{}{\vdash A, A^{\perp}}\ (identity) \qquad \frac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

Multiplicatives

$$\frac{}{\vdash 1}\ (one) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ (false)$$

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ (times) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ (par)$$

Additives

$(no\ rule\ for\ zero)$

$$\frac{}{\vdash \Gamma, \top}\ (true)$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ (left\ plus) \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ (with)$$

$$\frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ (right\ plus)$$

Exponentials

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ (of\ course) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ (weakening)$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ (dereliction) \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ (contraction)$$

$$\frac{}{\vdash A, A^{\perp}}\ (identity) \qquad \frac{\vdash \Gamma, A \quad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

Multiplicatives

$$\frac{}{\vdash 1}\ (one) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ (false)$$

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$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\ (with)$$

Exponetials ...

Linear negation

$$(X^{\perp})^{\perp} = X$$

$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$$

$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$

$$(A \mathbin{\&} B)^{\perp} = A^{\perp} \oplus B^{\perp}$$

$$(A \oplus B)^{\perp} = A^{\perp} \mathbin{\&} B^{\perp}$$

In this tutorial, we will focus on MLL

The sequent calculus derivation rules of MLL are:

$$\frac{}{\vdash A,\ A^{\perp}}(Ax) \qquad \frac{\vdash \Gamma,\ A \qquad \vdash \Delta,\ A^{\perp}}{\vdash \Gamma,\ \Delta}(Cut)$$

$$\frac{\vdash \Gamma,\ A \qquad \vdash \Delta,\ B}{\vdash \Gamma,\ A \otimes B,\ \Delta}(\otimes) \qquad \frac{\vdash \Gamma,\ A,\ B}{\vdash \Gamma,\ A \parr B}(\parr)$$

Linear negation is defined by :

$$X^{\perp\perp} = X$$

$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$

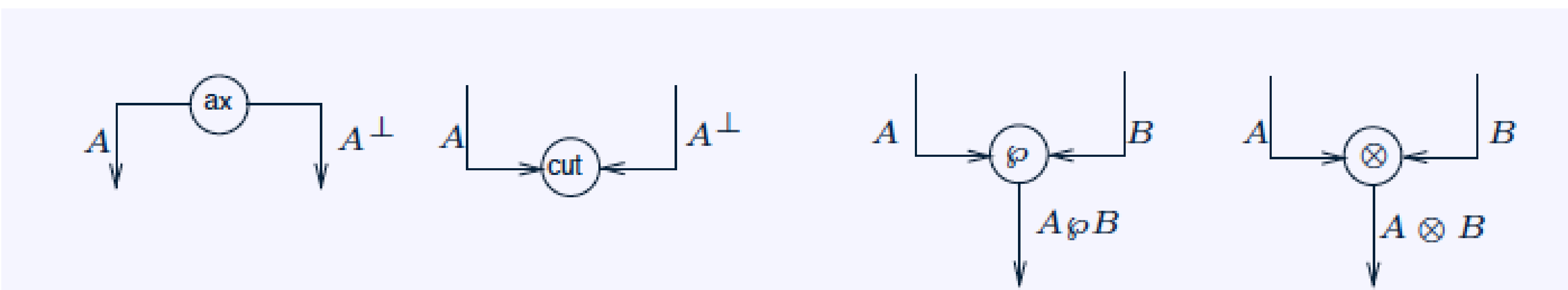
Proof Nets
A graph syntax for proofs

Proof structures

A **proof structure** is a directed acyclic graphs (d.a.g.) with pending edges (some edges have a source but no target)

- **nodes** (also called *links*) are labelled by one of the symbols ax, cut, $\parr$, $\otimes$ (corresponding to MLL rules).
- **edges** are typed by formulas of MLL.

according to the following typing rules:



For each node/link: ***premisses*** *= entering edges,* ***conclusions*** *= exiting edges*

Each edge is *conclusion* of a unique link and *premise* of at most one link.

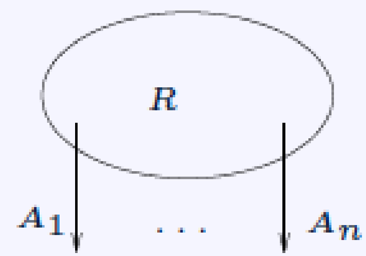
The pending edges are the conclusions of the proof structure.

Translation of proofs

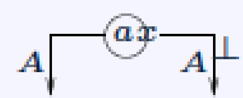
proof Π

translation $\Pi^* = R$

$$\frac{\Pi}{\vdash A_1, \ldots, A_n}$$

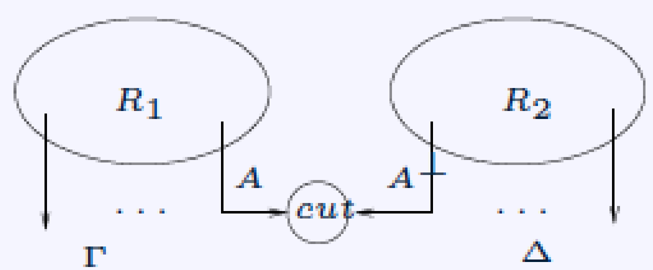


$$\frac{}{\vdash A, A^\perp}\ ax$$



$$\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, A} \qquad \dfrac{\Pi_2}{\vdash A^\perp, \Delta}}{\vdash \Gamma, \Delta}\ \text{cut}$$

$\Pi_i^* = R_i,\ i = 1, 2$

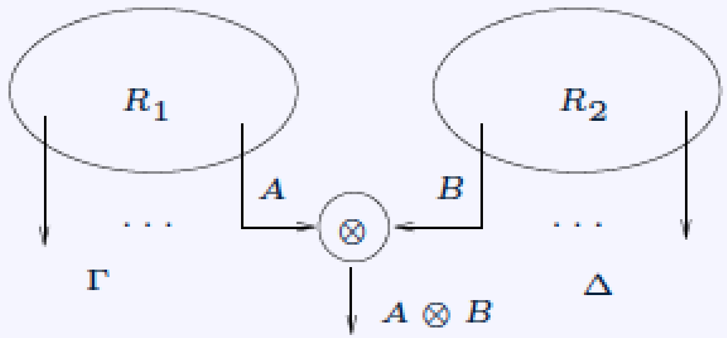


Translation of proofs (continued)

proof Π translation $\Pi^* = R$

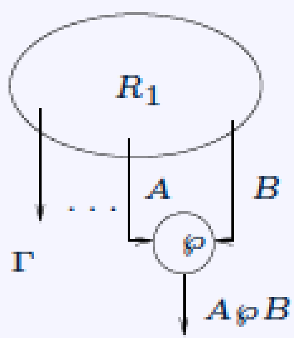
$$\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, A} \quad \dfrac{\Pi_2}{\vdash B, \Delta}}{\vdash \Gamma, A \otimes B, \Delta} \otimes$$

$\Pi_i^* = R_i, \ i = 1, 2$



$$\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, A, B}}{\vdash \Gamma, A \wp B} \wp$$

$\Pi_1^* = R_1$



example

Translate this sequent calculus proof. Start from axioms.... add links....

$$\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,ax \qquad \dfrac{}{\vdash B^{\perp}, B}\,ax}{\vdash B^{\perp}, A^{\perp}, A \otimes B}\,\otimes}{\vdash B^{\perp} \parr A^{\perp}, A \otimes B}\,\parr$$

Proof Nets

Definition 1 (Proof net). *A proof structure R is a proof-net if it is the image of a sequent calculus proof* (there exists a proof π s.t. $\pi^* = R$)

Internal condition!

Purely geometrical conditions (correction) characterize the proof structures which are proof nets

Theorem 1. *A proof structure is* ***correct*** *iff it is a* ***proof net****.*

Correctness criteria:

- (LT) Long trip [Girard]
- (AC) Connected-Acyclic [Danos-Regnier]
- ...

Definition 2 (Correctness criterion AC (Danos-Regnier)). *Let R be a proof structure.*

A switching s is a function on the nodes of R, which chooses, for each $\parr$-link, either the left or the right premise.

A proof structure R is correct if for each switching, the unoriented graph obtained by erasing for each $\parr$-link of R the edges not chosen by s. is:

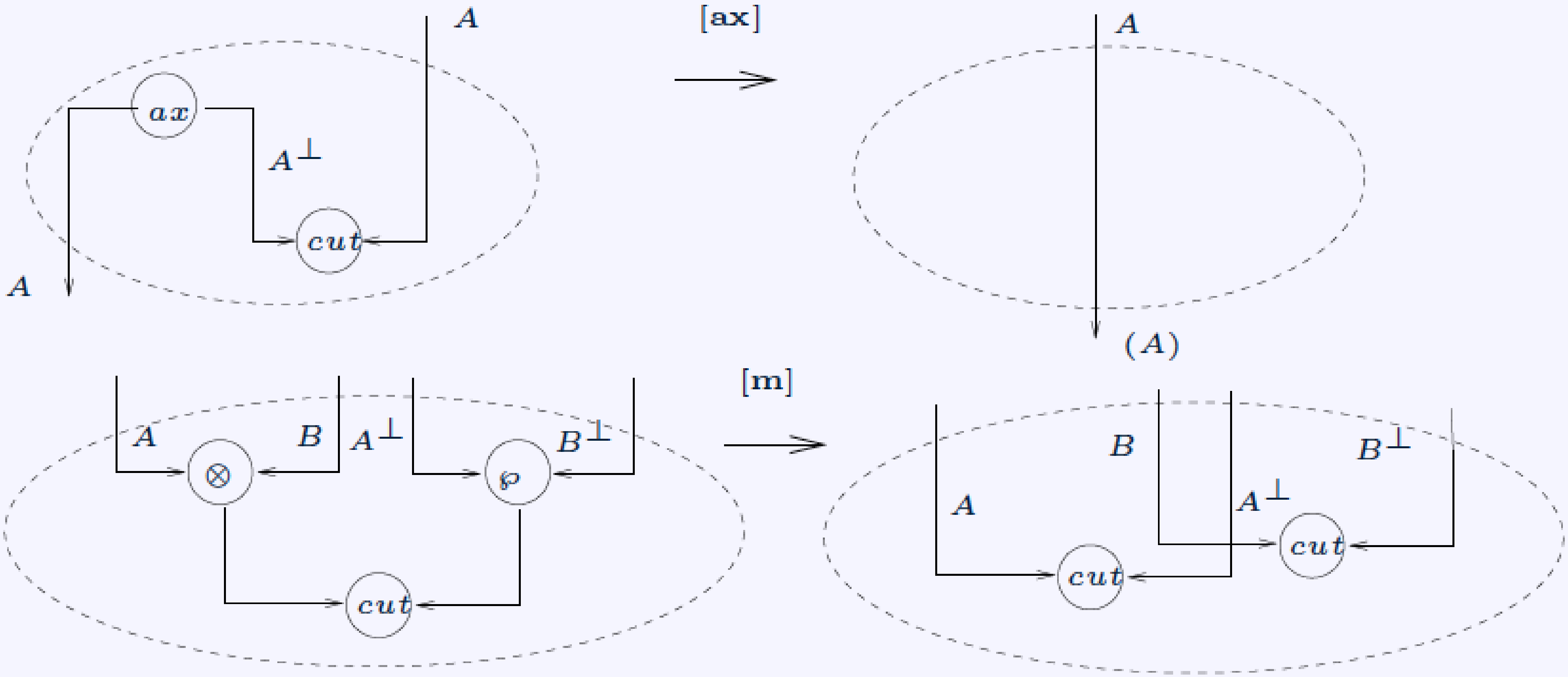
connected and acyclic

Correctness guarantees:

- ✓ Graph is image of a proof (sequentialization)
- ✓ Normalization terminates

The beauty of proof nets is **normalization**

Normalization (local graph reductions!)



MLL: properties of normalization

Lemma (preservation of correctness)

If the proof structure R is correct and reduces to R', then R' is correct.

Theorem (Strong normalization)

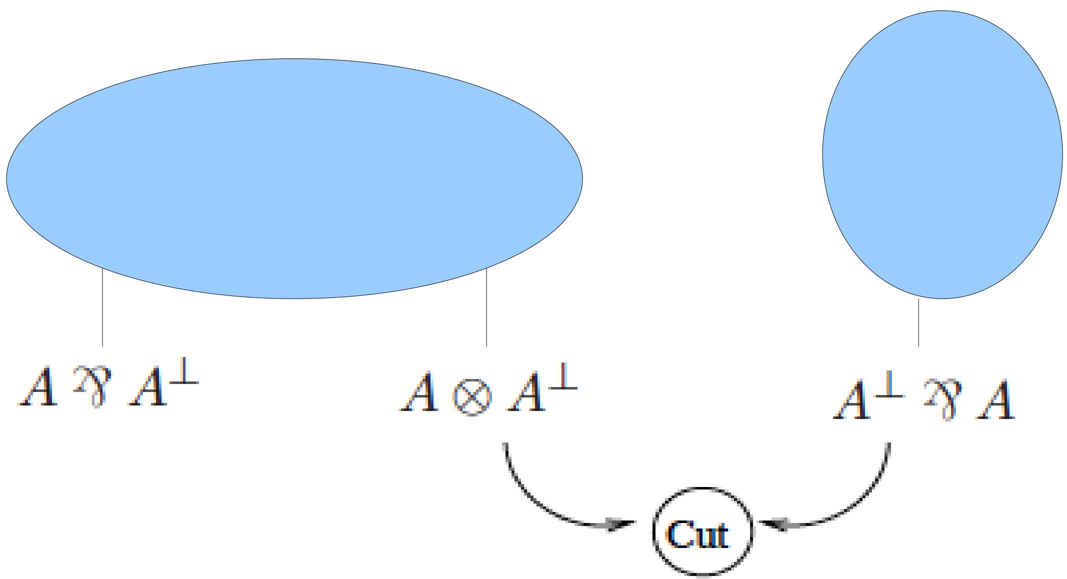
- *all cuts can be reduced*
- *the number of step to reach normal form is bound by the number of nodes*

Theorem (Confluence)

Normalization is confluent

Let us try out!

$$\frac{\vdash A \parr A^{\perp}, A \otimes A^{\perp} \qquad \vdash A^{\perp} \parr A}{\vdash A \parr A^{\perp}}\ cut$$



Write a proof net with this conclusion... and normalize it

How we write a proof net of these conclusions?

$A \parr A^{\perp}$ must type an edge conclusion of a par link, with premisses ....

$A \otimes A^{\perp}$ must type an edge conclusion of a tensor link, with premisses ....

Then we have to choose the axiom links!

Let us try one more. First, write a proof net with this conclusion...

$$(X \otimes X) \multimap (X \otimes X) =$$

$$(X \otimes X)^{\perp} \parr (X \otimes X) = (X^{\perp} \parr X^{\perp}) \parr (X \otimes X)$$

How we write a proof net? As before, all proof nets with the same conclusion, start with the same nodes (the formula tree!)
What distinguishes different proofs are the axiom links

To distinguish the different occurrences of atoms, let us write indices:

$$({X_1}^{\perp} \parr {X_2}^{\perp}) \parr (X_3 \otimes X_4)$$

In this case, we have two possible proofs, corresponding to two possible way to write axioms:
1,3 and 2,4
OR
1,4 and 2,3

parenthesis

In sequent calculus, they correspond to these two proofs (one uses exchange, one no)

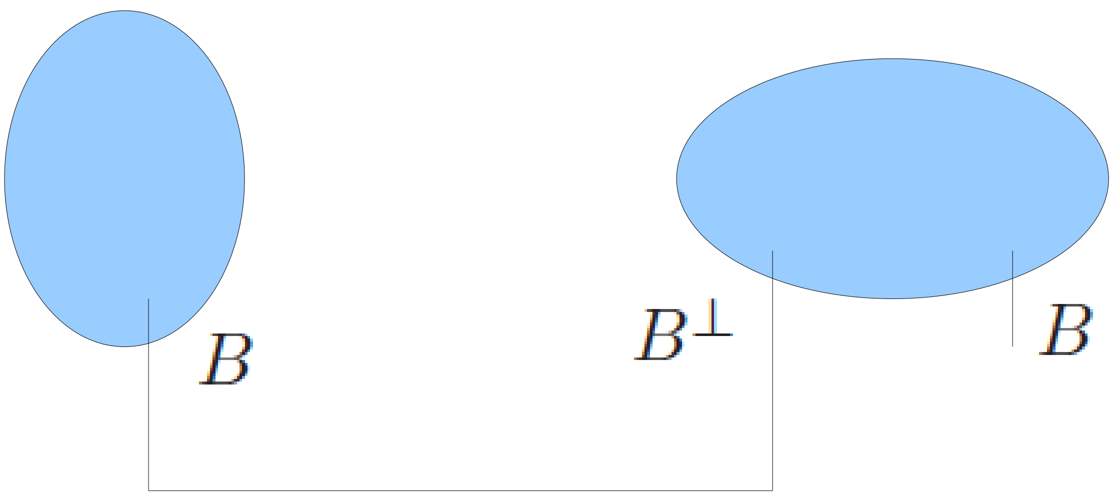
$$\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash {X_3}^{\perp}, X_1} \quad \cfrac{}{\vdash {X_4}^{\perp}, X_2}}{\vdash {X_1}^{\perp}, {X_2}^{\perp}, X_3 \otimes X_4}\ \otimes}{\vdash {X_1}^{\perp} \parr {X_2}^{\perp}, X_3 \otimes X_4}\ \parr}{\vdash (X_1 \otimes X_3)^{\perp} \parr (X_2 \otimes X_4)}\ \parr$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash {X_3}^{\perp}, X_2} \quad \cfrac{}{\vdash {X_4}^{\perp}, X_1}}{\vdash {X_1}^{\perp}, {X_2}^{\perp}, X_3 \otimes X_4}\ \otimes}{\vdash {X_1}^{\perp} \parr {X_2}^{\perp}, X_3 \otimes X_4}\ \parr}{\vdash (X_1 \otimes X_3)^{\perp} \parr (X_2 \otimes X_4)}\ \parr$$

When we have a formula whose normal proofs are exactly two, we have a good candidate to code BOOLEANS :)

Let us indicate the formula $(X^\perp \parr X^\perp) \parr (X \otimes X)$ with B (for boolean).
We call one proof ***true***, and the other ***false***...

We can feed one of our two values to a proof which takes a boolean, and return a boolean.



We know that the normal form (i.e the result of computation) will be of type B... Hence one of our two values.

Try to normalize one of the proofs of $(X_1{}^{\perp} \parr X_2{}^{\perp}) \parr (X_3 \otimes X_4)$

with the proof net which has conclusions

$$(X_1 \otimes X_2) \otimes (X_3{}^{\perp} \parr X_4{}^{\perp}) \qquad (X_5{}^{\perp} \parr X_6{}^{\perp}) \parr (X_7 \otimes X_8)$$

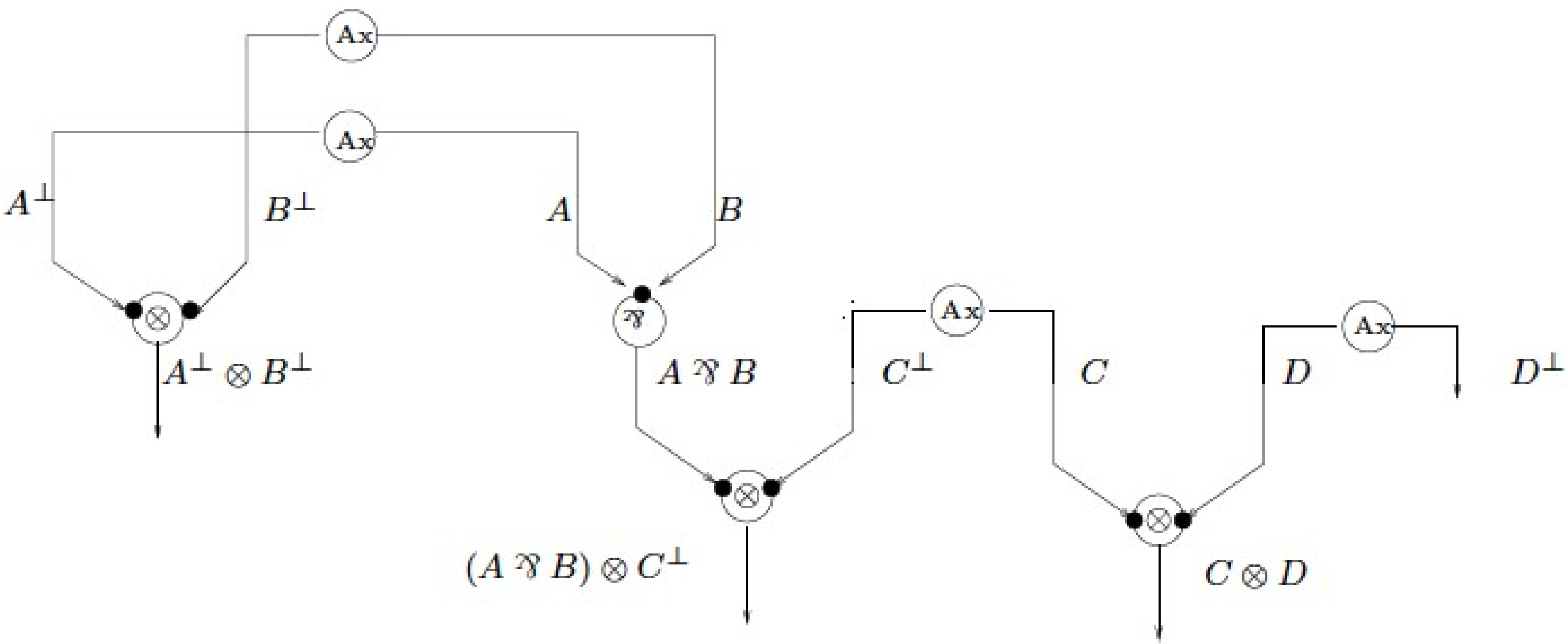
and axiom links:

1,6
2,5
3,7
4,8

What is the function coded by this proof net?

Sequentialization

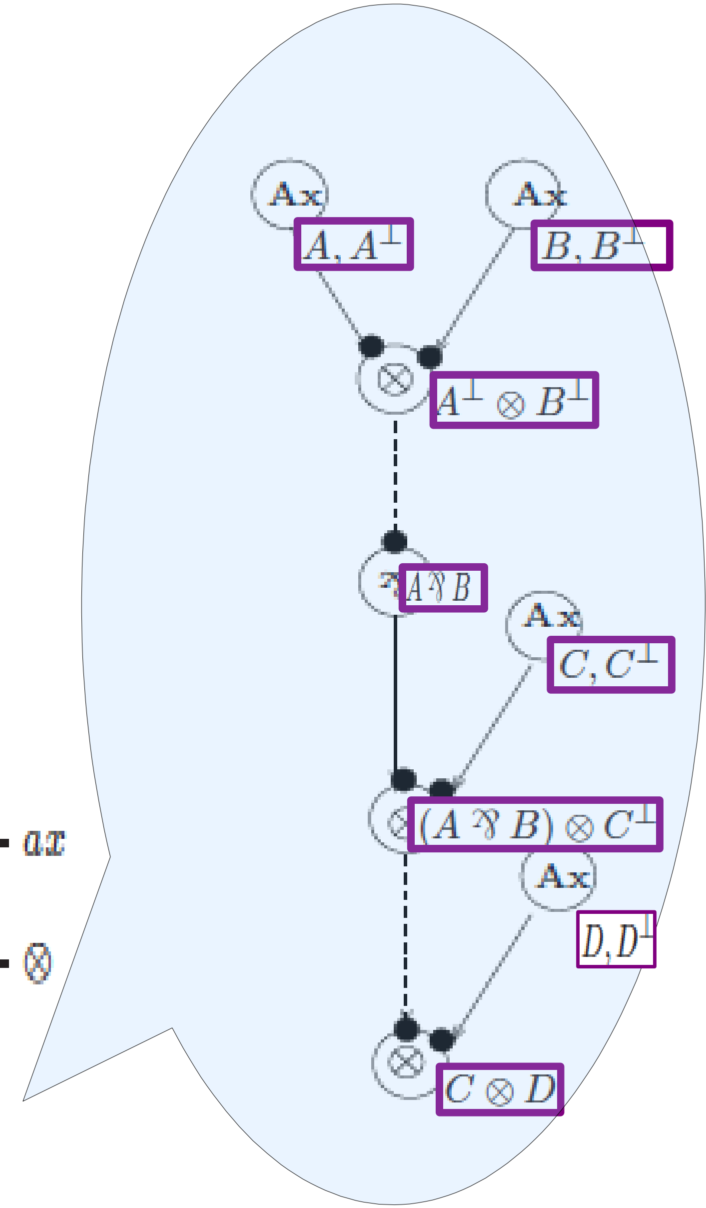
We have a proof net. The problem: it is the image of a sequent calculus proof? And which?

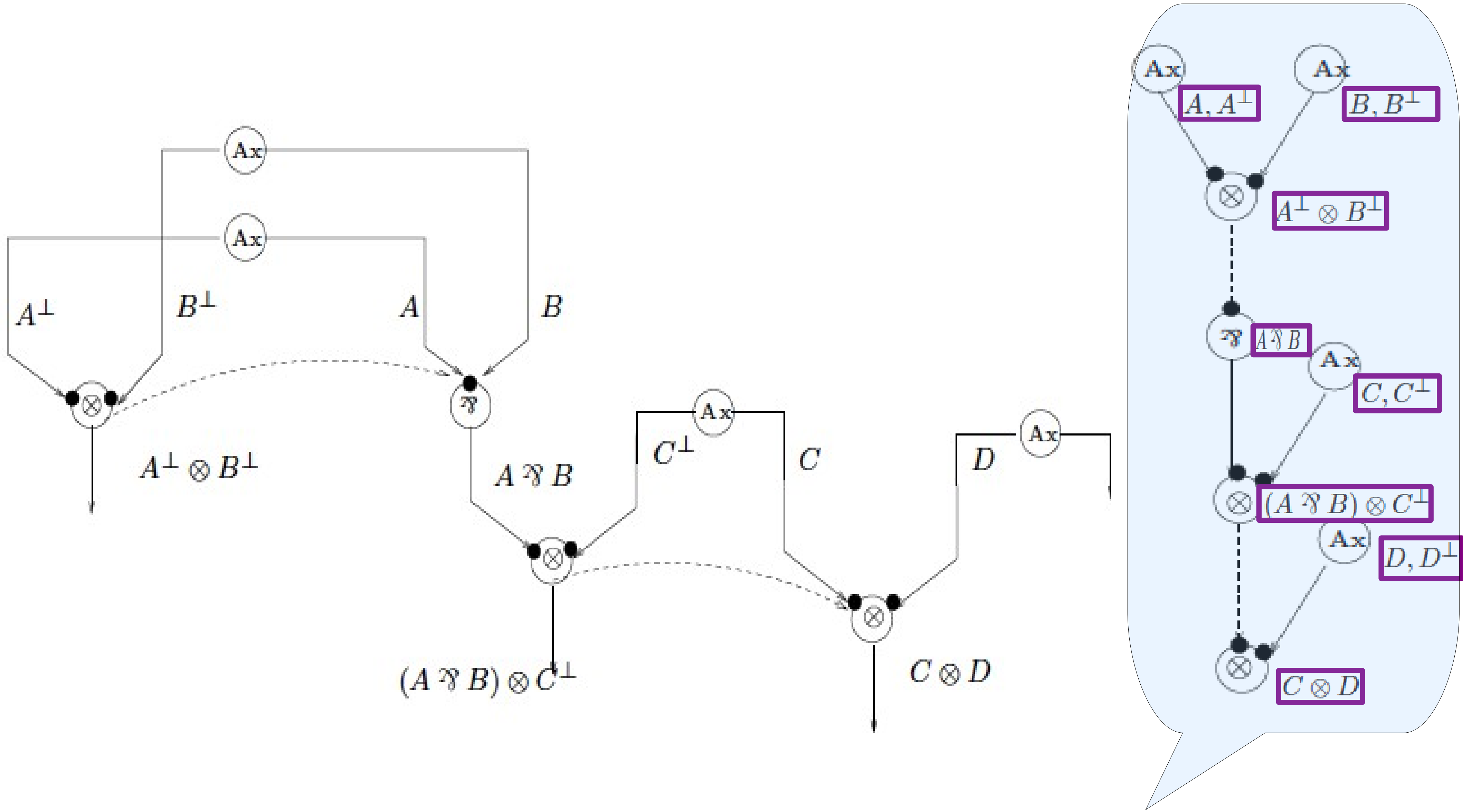


In fact, what is a sequent calculus proof?

A sequent calculus proof is a tree of rules...

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash A, A^\perp$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash B, B^\perp$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash A^\perp \otimes B^\perp, A, B$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash A^\perp \otimes B^\perp, A \parr B$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash C, C^\perp$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash D, D^\perp$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C \otimes D, D^\perp,$}
\end{prooftree}
$$





We added one untyped edge between the $\parr$ link and the leftmost $\otimes$ link and one untyped edge between the middle $\otimes$ and the rightmost one.

Now consider the partial order induced by the directed graph

Memo:

a directed acyclic graph (dag) G is
an oriented graph without (oriented) cycles.

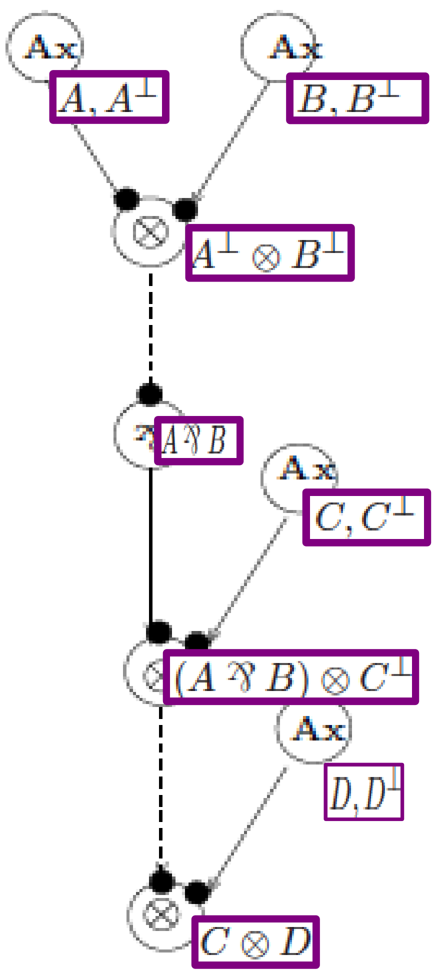
The **transitive closure** induces
a **strict partial order** on the nodes of G:

$a < b$ iff $a \leftarrow b$ (= there is an edge from b to a)

The **skeleton** of a dag G is the graph that has
the same vertices as G and
whose edges are the edges of G which are not transitive

(it is the canonical representation of the partial order)

Such a tree directly corresponds to the following sequent calculus proof:



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,ax \qquad \dfrac{}{\vdash B, B^\perp}\,ax}{\vdash A^\perp \otimes B^\perp, A, B}\,\otimes}{\vdash A^\perp \otimes B^\perp, A \parr B}\,\parr \qquad \dfrac{}{\vdash C, C^\perp}\,ax}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C}\,\otimes \qquad \dfrac{}{\vdash D, D^\perp}\,ax}{\vdash A^\perp \otimes B^\perp, (A \parr B) \otimes C^\perp, C \otimes D, D^\perp,}\,\otimes$$

Note: the original proof of sequentialization [Girard] uses empires

Di Giamberardino-Faggian, APAL08

IDEA

We sequentialize a proof net by adding to it enough untyped edges (**sequential edges**) to retrieve a sequent calculus proof.

Sequential edges are a generalization of jumps [Girard 91,96: quantifiers, additives]

We add edges with the only constraint that the proof net remains correct. The net is saturated when any additional edge

– either does not increase the order,
– or violates the correctness criterion.

Which **correctness** ?

Parenthesis: if we focus on acylicity, Danos-Regnier criterion can be reformulated (in equivalent way)

Let R be a proof structure; a *switching path* of R is a path which does not use any two edges entering on the same $\parr$ link (such edges are called *switching edges*); a *switching cycle* is a switching path which is a cycle.

Definition 3 (Correctness criterion). *A proof structure is correct if it does not contain any switching cycle.*

$$\frac{}{\vdash A,\ A^{\perp}}(Ax) \qquad \frac{\vdash \Gamma,\ A \qquad \vdash \Delta,\ A^{\perp}}{\vdash \Gamma,\ \Delta}(Cut)$$

$$\frac{\vdash \Gamma,\ A \qquad \vdash \Delta,\ B}{\vdash \Gamma,\ A \otimes B,\ \Delta}(\otimes) \qquad \frac{\vdash \Gamma,\ A,\ B}{\vdash \Gamma,\ A \parr B}(\parr)$$

$$\boxed{\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}(Mix)}$$

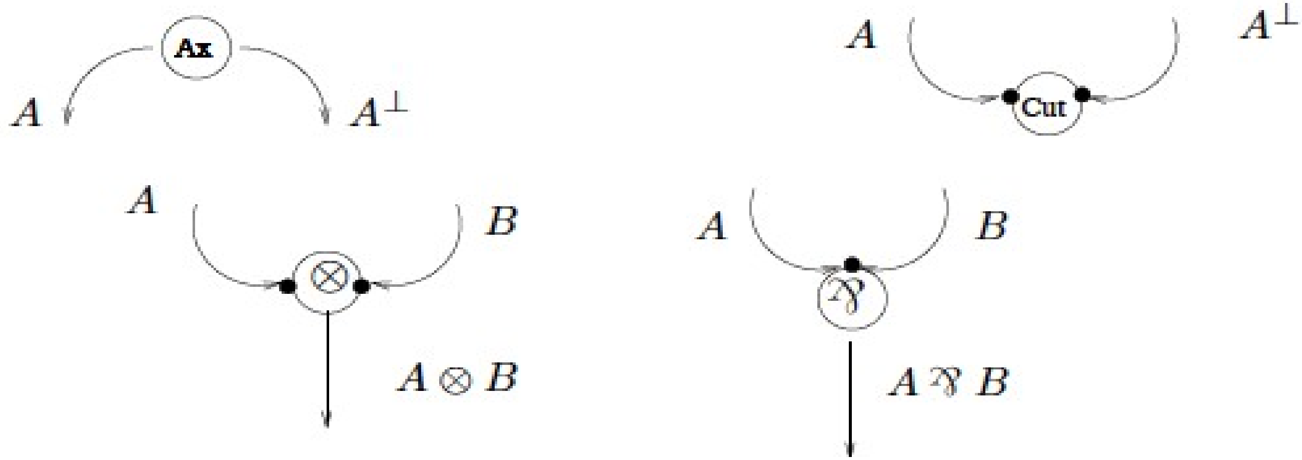
we can throw away MIX later
By requiring **connectness**

To accommodate additional edges,
we proceed in two steps:

1. partition the entering edges of a node into ports
2. add edges

1. **Ports associated to a link** To each link of a proof structure R, we associate a partition of *all* entering edges into ports, in the following way:

- a $\parr$ link of conclusion $A \parr B$ has *only one port*, containing both A, B
- a $\otimes$ (resp. *cut*) link of conclusion $A \otimes B$ and premises A, B (resp. $A, A^{\perp}$), has *two ports*, one containing A, one containing B (resp. $A^{\perp}$) .



2. add untyped edges; each edge which enters a node choose a port.

The definition of switching can now be reformulated:

Definition 4 (Switching path). *A* switching path *of a proof structure R is a path which does not use any two edges entering trough a same port.*

Given a structure so enriched, it is correct if it has no switching cycles

Definition 4 **(Switching path).** *A* switching path *of a proof structure R is a path which does not use any two edges entering trough a same port.*

Syntax revised- Now:

A proof net is a special case of (without sequential edges).
A sequent calculus proof of MLL, essentially is also a special case
(there are enough sequential edges to recover the tree-like structure of the proof).

We sequentialize a proof net by adding to it enough untyped edges (sequential edges) to retrieve a sequent calculus proof.

We add edges with the only constraint that the proof net remains correct.

The net is saturated when any additional edge

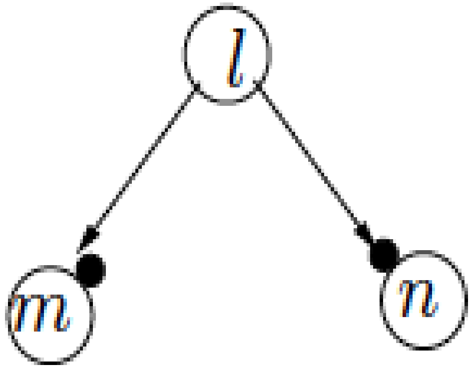
- either does not increase the order,
- or violates the correctness criterion.

Lemma 1 (Arborisation).
If R is saturated then $\prec_R$ is arborescent.

- Any net can be saturated.
- The order associated to a saturated net is arborescent.
- If the order $\prec_R$ associated to R is arborescent, then $Sk(R)$ is a forest and we can associate to R a proof π^R in the sequent calculus.

Lemma 1 (Arborisation).
If R is saturated then $\prec_R$ is arborescent.

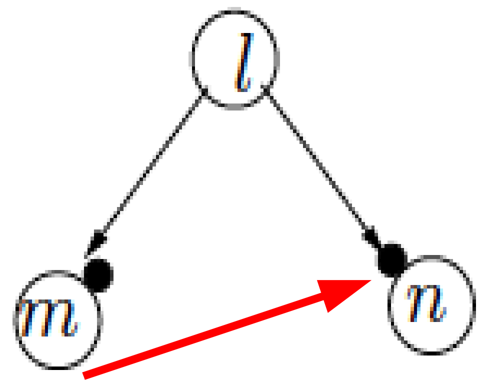
Proof. We prove that if $\prec_R$ is not arborescent, then there exist two links m and n s.t. adding a sequential edge between m and n (or viceversa) doesn't create switching cycles and makes the order increase.



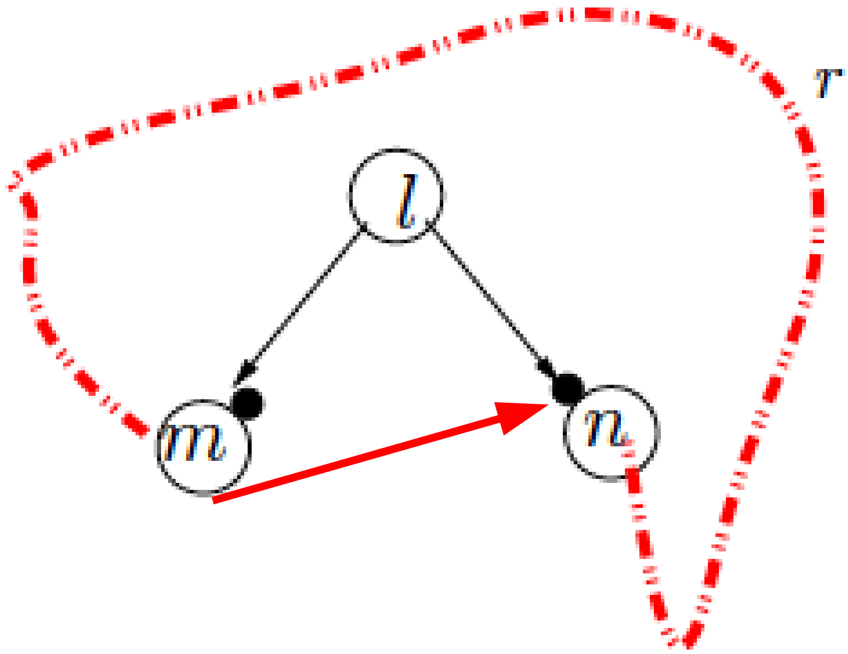
MEMO**. A strict order is **arborescent** = each element has **at most one predecessor

If <_R is not arborescent, there is a link l, with two (incomparable) predecessor

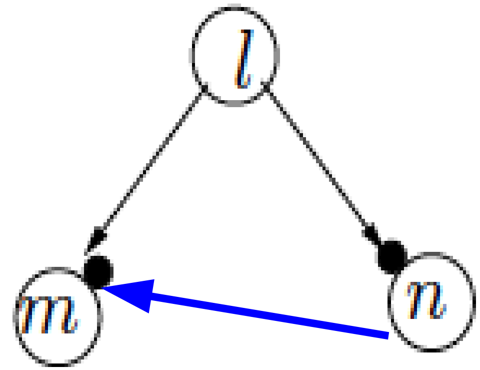
Add an edge m-->n:
the order increases



But creates a cycle !?!

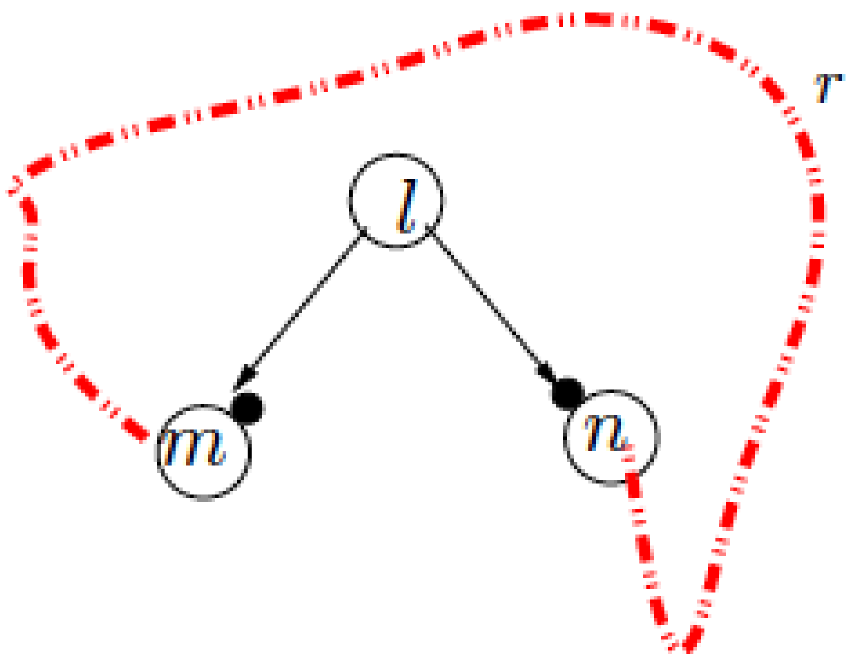


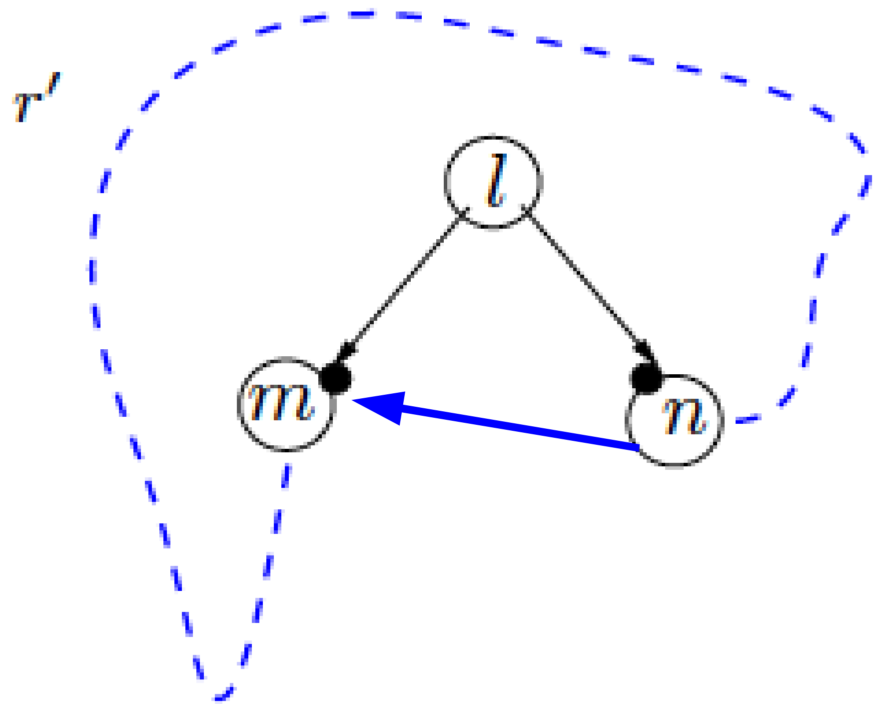
Means there is a switching
path **r** between n and m



Add an edge n-->m:
the order increases

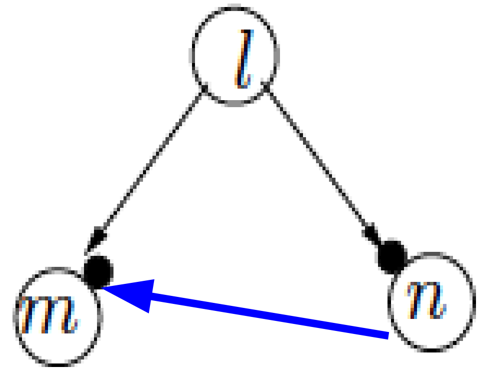
But creates a cycle !?!





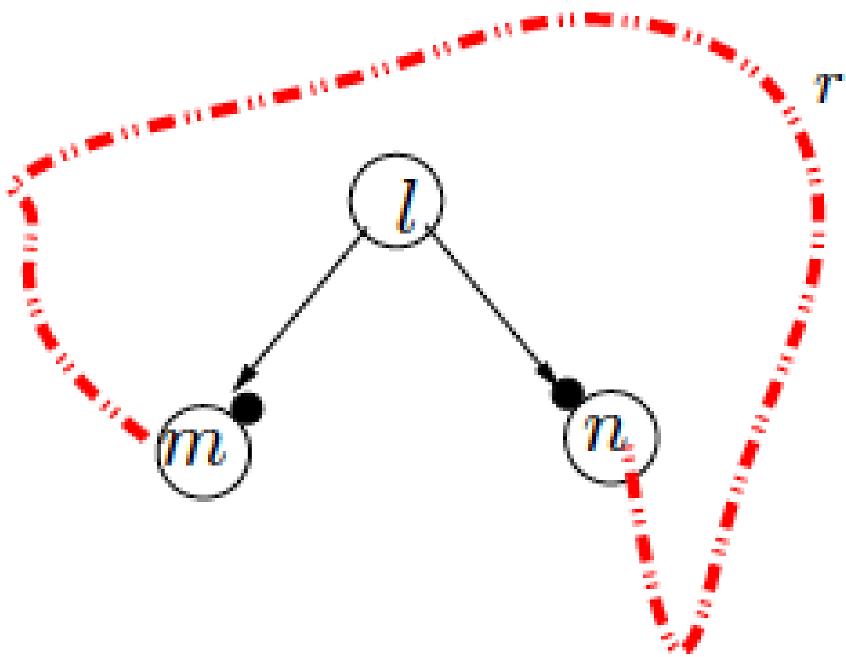
Means there is a switching path **r** between n and m

means in R there is a switching path **r'** between m and n

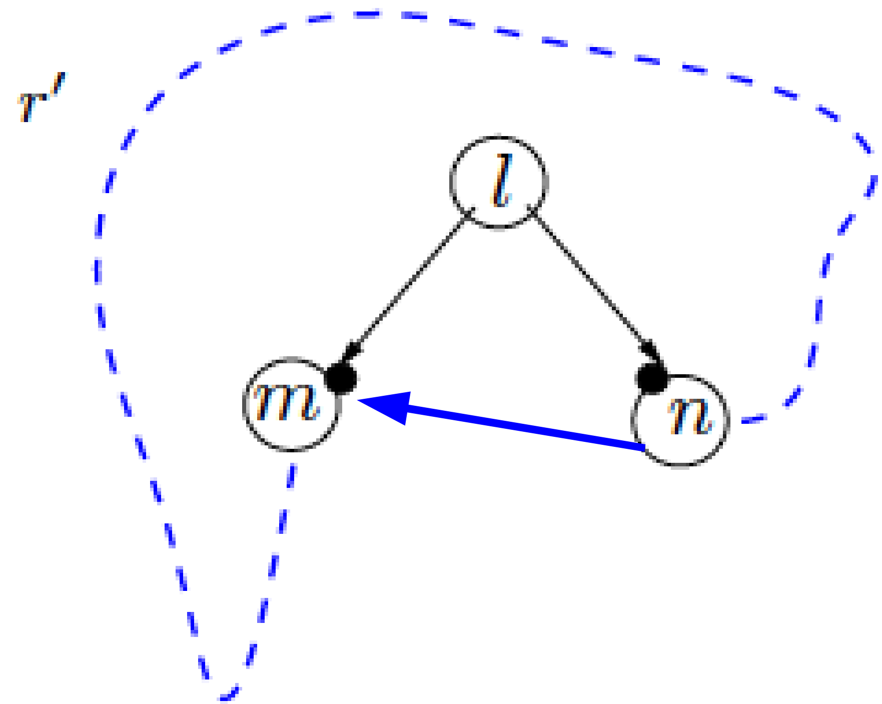


Add an edge n-->m:
the order increases

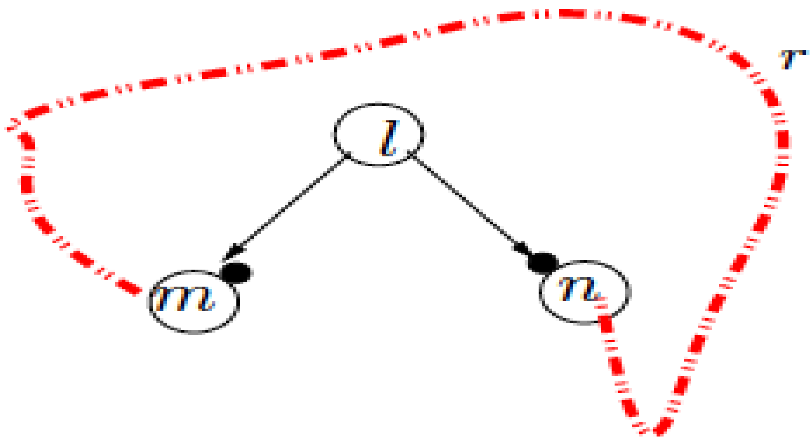
But creates a cycle !?!



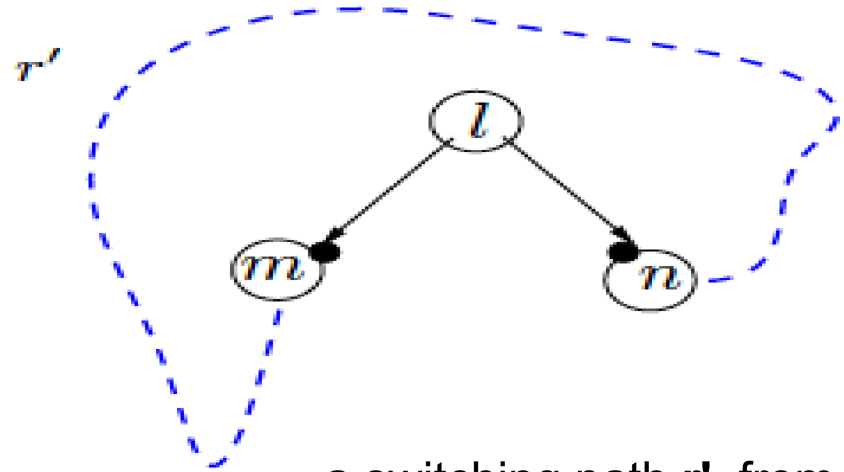
Means there is a switching
path **r** between n and m



means in R there is a switching
path **r'** between m and n



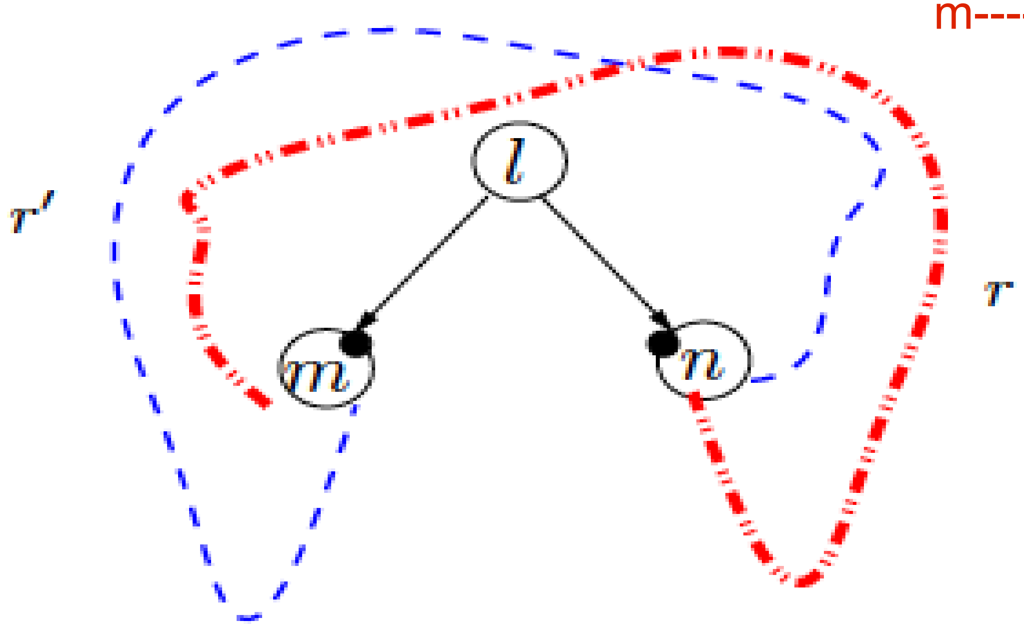
switching path **r** from m to n



a switching path **r'** from n to m

Case 1: r and r' do not have 2 edges which belong to the same port

m----->n + n------>m



Follow r from m to n. Let c be the first node whose port z contains both an edge of r and an edge of r'.

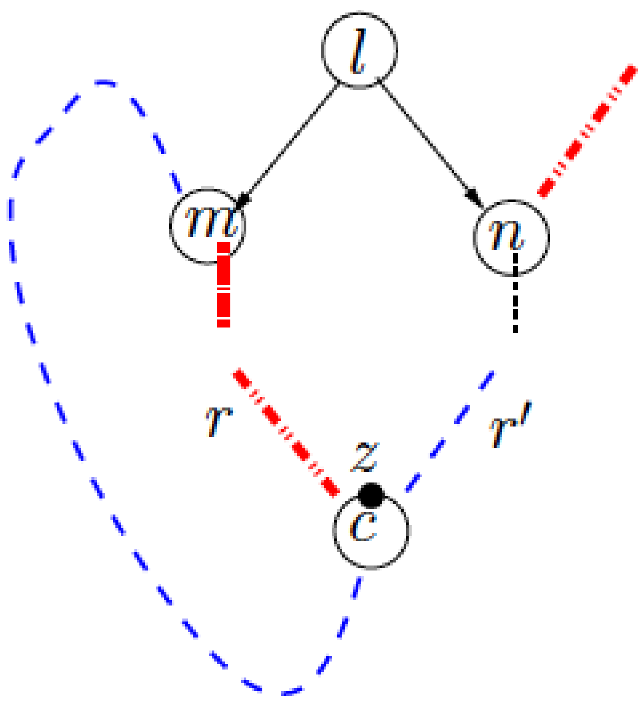
Follow r' from n to c.

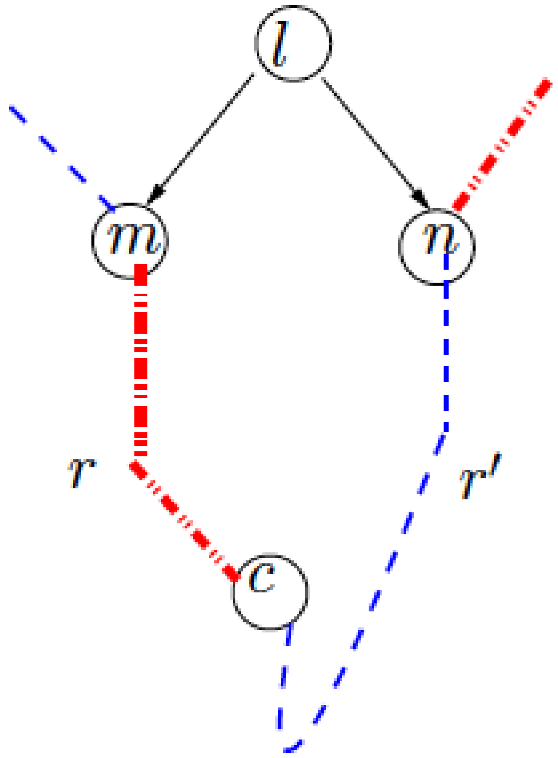
Case 2a: r' enters in z

m----->c + c ------>m

Case 2b: r' enters in a different port

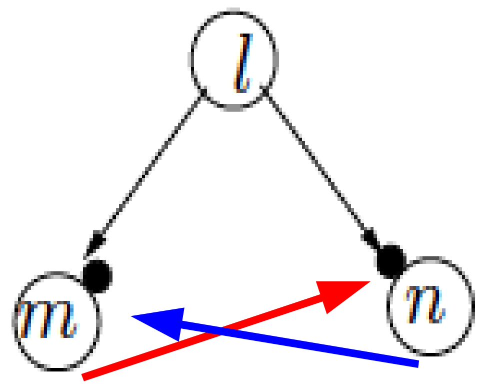
m----->c + c ------>n + n <-- l + l -->m





Moral:

if the order is not a tree there is such a configuration:



and we can always add an edge to increase the order
(while preserving correctness)