Deep into optimality

Complexity and correctness of shared implementation of bounded logics

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Joint work with Stefano Guerrini and Thomas Leventis

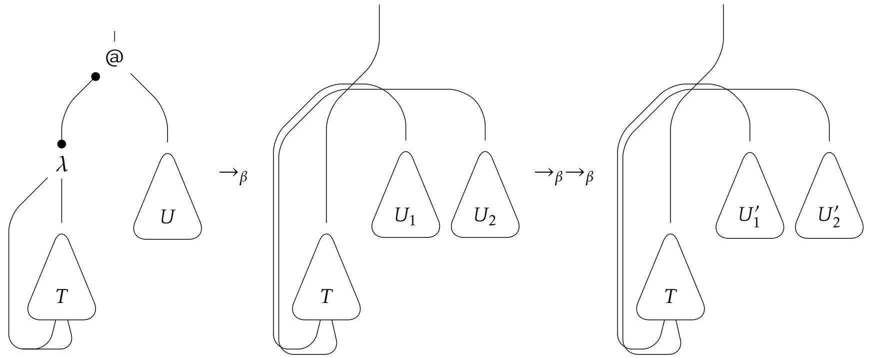
Outline

Lévy's program [Lév78]

Goals for λ-calculus:

1. best recursive reduction strategy
2. cost model of reduction

Example: reduction with duplication



Lévy's proposal

Idea:

- reduction can create duplicated redexes
- group duplicated redexes in *families*

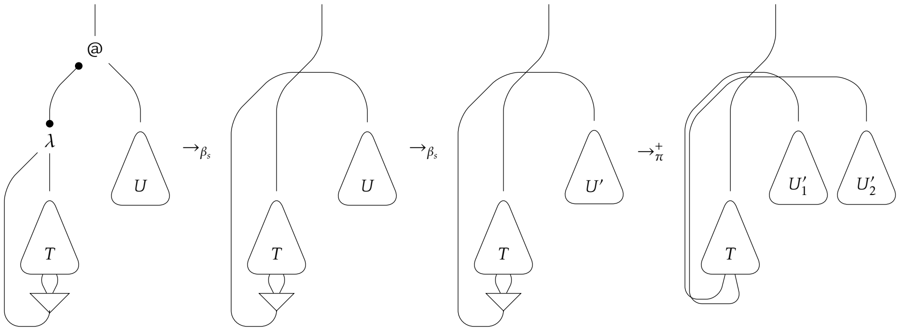
Solution:

1. cost: family as cost unit
2. strategy: lazy and parallel, call family by need (unimplemented)

Concrete implementation: sharing graphs [Lam89]

Graph rewriting system, permitting sharing of contexts

Example: shared reduction



- dissection of β-reduction
- model for linear logic proof-nets

Sharing graphs implements a Lévy-optimal reduction

Cost: redex families?

Theorem (Families and complexity, [AM98])

For some n-sized λ-terms shared reduction needs to reduce a number of families in

$$\mathcal{O}(n)$$

while every TM-equivalent implementation takes

$$\Omega(K(n,l))$$

Redex families are not a valid cost model for λ-calculus

Inefficiency of shared reduction?

Misconception: shared reduction is inefficient

Theorem (Statman, 1979)

Every concrete reduction of n-sized simply typed λ-terms takes

$$\Omega(K(n,l))$$

Where does the complexity discrepancy between families and actual cost come from?

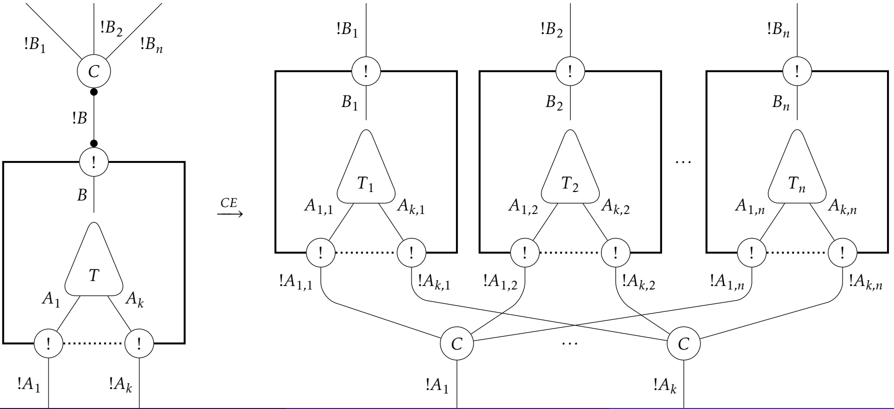
- oracle, managing levels indexes
- duplication itself [ACM04]

Is shared reduction convenient?

Bounded logics (EAL, LAL, ...)

- linear logic variants, with bounds on recursion
- intrinsic complexity characterization (elementary, polynomial)
- C-H correspondence: proof-net as syntax tree

Example: box duplication



Shared implementation and bounded logics

Property (Stratification in elementary and light logics)

Boxing depth is invariant w.r.t. reduction.

Shared implementation

- no need of re-indexing work (the oracle)
- complexity easier to study

Theorem (Complexity [BCD11])

Shared normalization of n-sized typed λ-terms takes

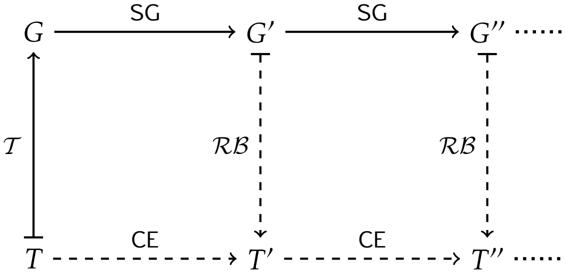
$$lenght = \begin{cases} \mathcal{O}(K(k,n)) & \text{if in EAL} \\ \mathcal{O}(pol_k(n)) & \text{if in LAL} \end{cases}$$

Theorem (Soundness and completeness [BCD11])

Shared normalization of EAL/LAL typed λ-terms is sound and complete w.r.t. proof-nets normalization.

Semantic approach [BCD11]

$\mathcal{T}$: initial translation
$\mathcal{RB}$: semantical readback

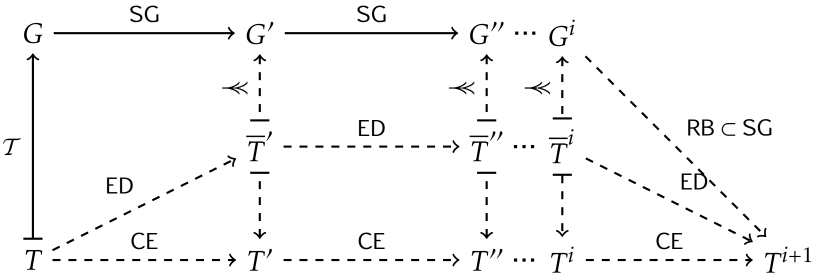


Syntactical approach [Gue99, GMM03]

RB: syntactical readback, by means of reduction

ED: intermediate system, with Explicit Duplication work

Simulation relationship between shared and standard reduction



Reduction costs, overview

Cut elimination or β-reduction

- duplication performed box by box
- non-local style
- cost of $N \to N'$: $|N| - |N'|$

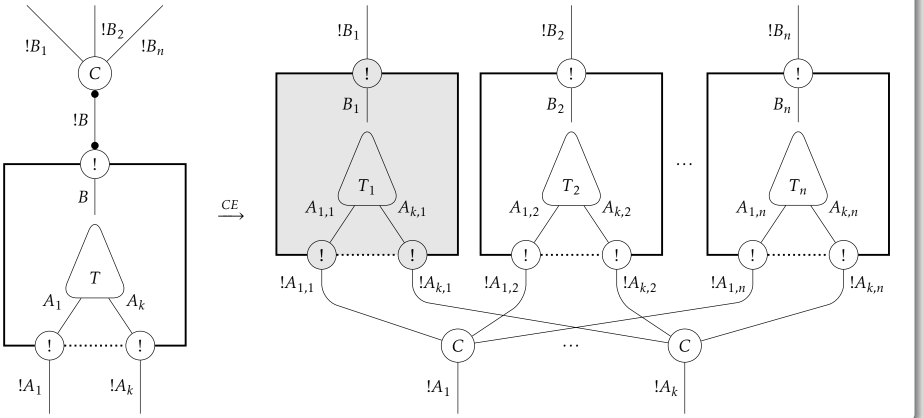
Shared reduction

- duplication performed node by node
- local style
- cost of $G \to G'$: k

Reduction costs, insight

Cost of a step r of box duplication

Example: propagation of n-ary contractor node



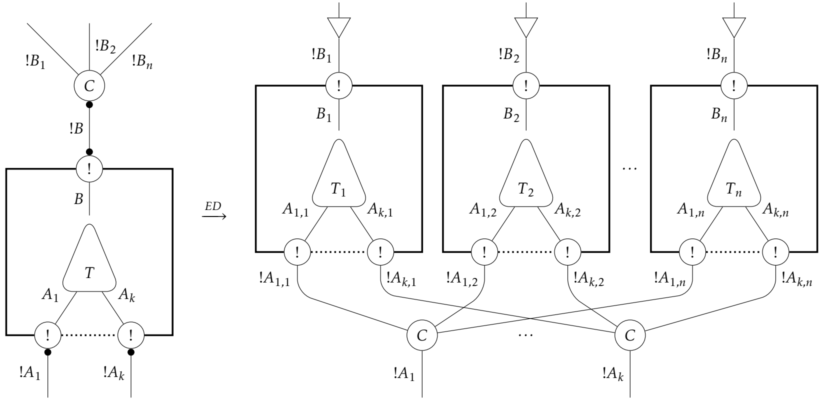
Cost: $\mathcal{C}(r) = |N| - |N'| = (n-1)|T| + c$

Half-way meet

Idea: intermediate system, between standard and shared reduction

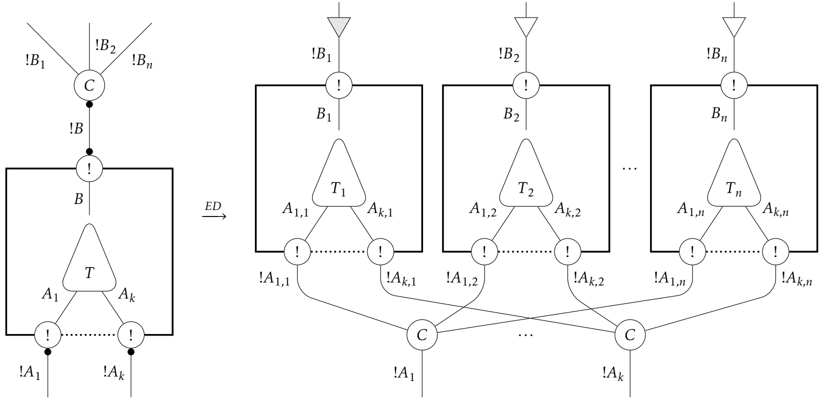
- outside: global duplication style from standard reduction
- inside: local markers propagation from shared reduction

Example: duplication with markers



Standard-intermediate side, intuition

Example: duplication and marker propagation lenght



Cost of sequence s: $\mathcal{C}(s) = \text{steps}(s) - \text{shadow-steps}(s)$

Every duplication step introduces: cost $\leq (n-1)|T| + c$

Standard-intermediate side, claim

Lemma

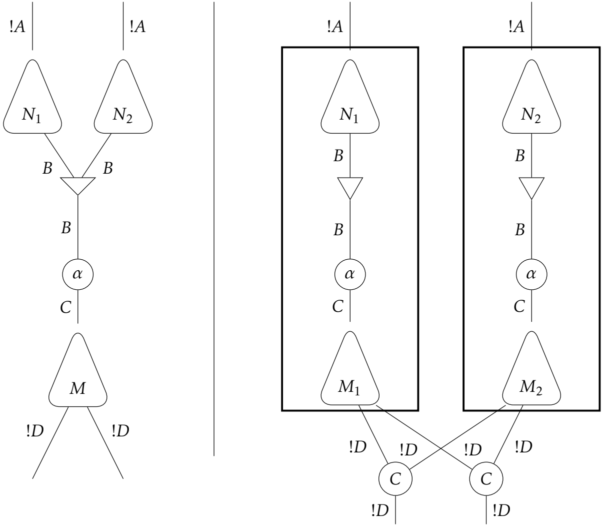
For any intermediate reduction exists a corresponding standard reduction such that:

- *they converge to the same λ-term*
- $\mathcal{C}_{CE} \geq \mathcal{C}_{ED}$

Standard reduction cost translated into intermediate rewriting steps

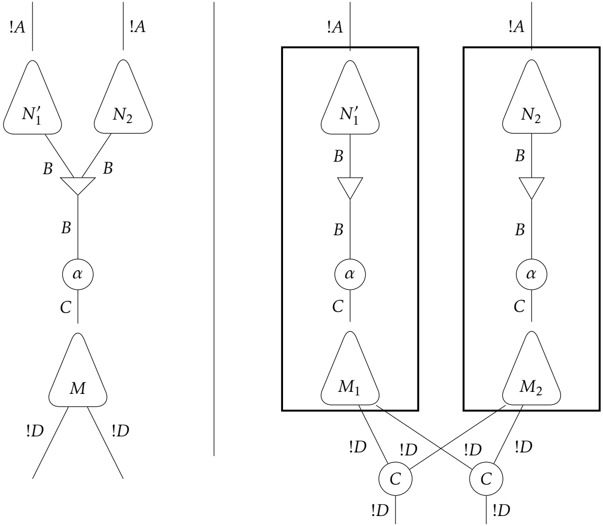
Shared-intermediate side, intuition (1 of 4)

Example: reduction in a non-shared area



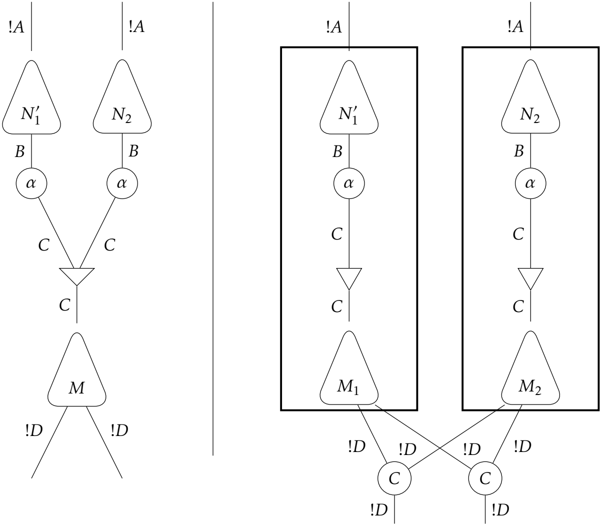
Shared-intermediate side, intuition (2 of 4)

Example: fan propagation



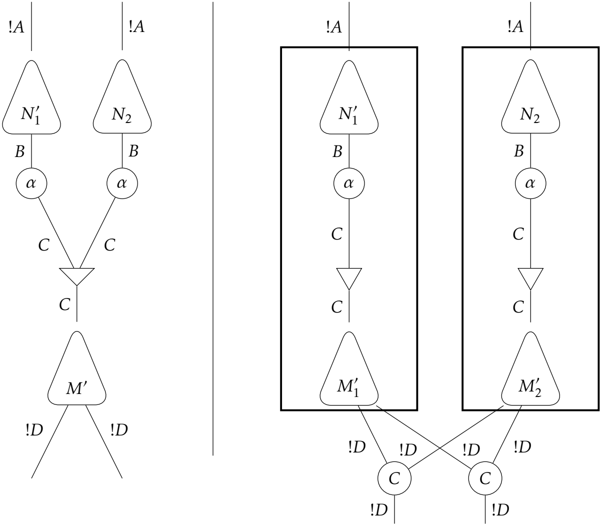
Shared-intermediate side, intuition (3 of 4)

Example: reduction in a shared area



Shared-intermediate side, intuition (4 of 4)

Example: end

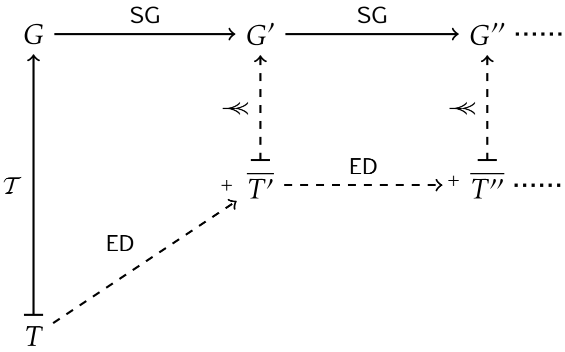


Shared-intermediate side, claim

Lemma (Simulation)

Any shared reduction can be simulated by an intermediate reduction sequence such that

- $\mathcal{C}_{ED} \geq \mathcal{C}_{SG}$



Final conjecture

From previous lemmas

- $CE \Leftarrow ED \Leftarrow SG$
- $\mathcal{C}_{CE} \geq \mathcal{C}_{ED} \geq \mathcal{C}_{SG}$

Theorem (Soundness and complexity)

For any shared reduction, exists a corresponding standard reduction such that:

- *they converge to the same λ-term*
- $\mathcal{C}_{CE} \geq \mathcal{C}_{SG}$

Moreover

- optimality comes from the best sharing reductions
- with non-linear terms $\mathcal{C}_{CE} > \mathcal{C}_{SG}$

Aimed contributions

New results on optimal reduction of bounded logic terms

- stronger result for complexity
- stronger result of correctness
- new syntactical proof approach to complexity problem

Open problems

- relationship of sharing graphs and other systems
  e.g. latest GoIs
- complexity of optimal reduction in the general case
  (optimality is optimal, indeed)

Thank you.

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