Strong bounds for light linear logic by levels

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Introduction

Implicit computational complexity

Capture a complexity class by syntactic restriction on a model of computation

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Why?

Bound on the execution of a program

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- The model of computation :
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- The complexity class : Polynomial time
- The model of computation : ~~λ calculus~~ LL proof nets
- The syntactic restriction : ~~type system~~ LL subsystem

1. Linear logic and complexity
   - Linear logic
   - Origins of complexity
2. Existing systems
   - LLL
   - L^4
   - L_0^4
3. Context semantics (Dal Lago 2006)
   - A notion of future duplicates
   - Capturing "duplicates" using paths
4. Strong bound for LLL, L^4, L_0^4
   - Strong bound LLL
   - Strong bound for L^4
   - Strong bound for L_0^4
   - Strong bound $DLALL_0$

System F

Types in system F

$$A, B ::= X \mid A \Rightarrow B \mid A \wedge B \mid \forall X, A \mid \exists X, A$$

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$$\underline{n} \qquad \lambda f.\lambda x.\ \underbrace{f(f(\dots(f\ x)))}_{n \text{ applications of } f} : \mathbb{N}$$

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$+1$	$\lambda n.\lambda f.\lambda x.n\ f(f\ x) : \mathbb{N} \Rightarrow \mathbb{N}$
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Typable in system F $\Rightarrow$ strongly normalizes

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Which complexity ?

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$$\begin{array}{ll} \mathbb{N} & \forall X, (X \Rightarrow X) \Rightarrow X \Rightarrow X \\ \underline{n} & \lambda f.\lambda x.\ \underbrace{f(f(\dots(f\ x)))}_{n \text{ applications of } f} : \mathbb{N} \\ +1 & \lambda n.\lambda f.\lambda x.n\ f(f\ x) : \mathbb{N} \Rightarrow \mathbb{N} \\ exp & \lambda n.n\ \underline{2} : \mathbb{N} \Rightarrow \mathbb{N} \end{array}$$

Which complexity ? We need a refinement

Linear logic

LL formulas

$$A, B ::= X \mid \qquad A \Rightarrow B \mid A \wedge B \mid \exists X, A \mid \forall X, A$$

Linear logic

LL formulas

$$A, B ::= X \mid \quad !A \multimap B \mid A \wedge B \mid \exists X, A \mid \forall X, A$$

$$A \Rightarrow B := !A \multimap B$$

Linear logic

LL formulas

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Ressources, *a priori* not duplicable

$\forall A, A \multimap (A \wedge A)$ ✗

dereliction	$\forall A, !A \multimap A$	✓
duplication	$\forall A, !A \multimap (!A \wedge !A)$	✓
promotion	$A \vdash B \rightsquigarrow !A \vdash !B$	✓
digging	$\forall A, !A \multimap !!A$	✓

Linear logic

LL formulas

$$A, B ::= X \mid \quad !A \mid \quad A^{\perp} \vee B \mid A \wedge B \mid \exists X, A \mid \forall X, A$$

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Linear logic

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Linear logic

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$$A, B ::= X \mid X^{\perp} \mid !A \mid ?A \mid A \parr B \mid A \otimes B \mid \exists X, A \mid \forall X, A$$

$$A \Rightarrow B := !A \multimap B \qquad A \multimap B := A^{\perp} \parr B$$

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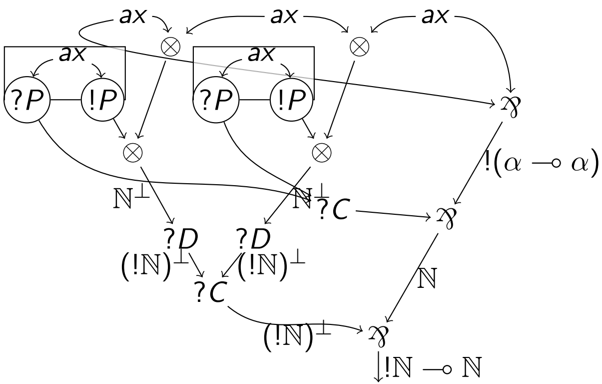
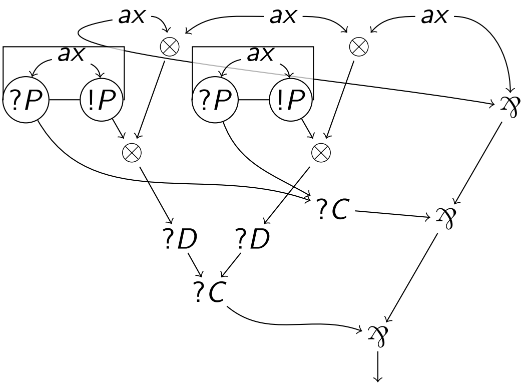
Proof net example : $\lambda n.\lambda f.\lambda x.(nf)(nfx)$



Proof net example : $\lambda n.\lambda f.\lambda x.(nf)(nfx)$



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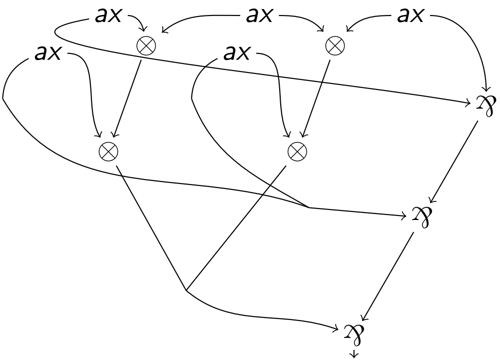

Proof net $\simeq$ syntactic tree + information on duplication

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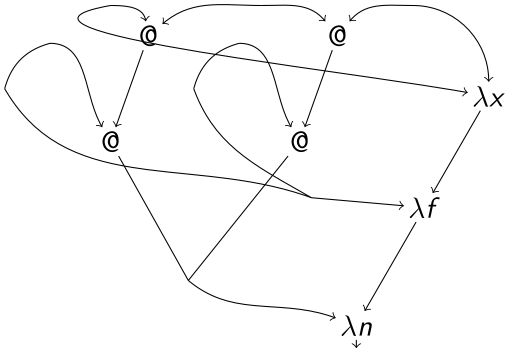

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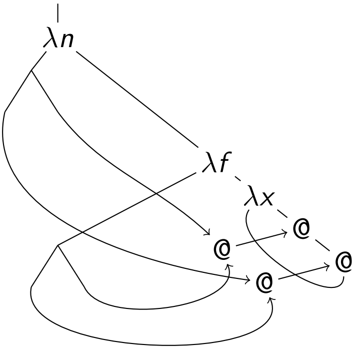
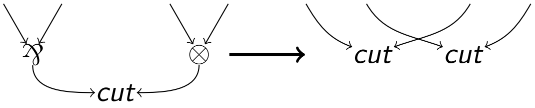
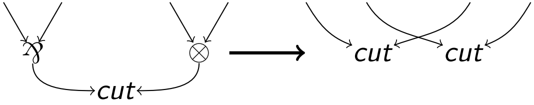


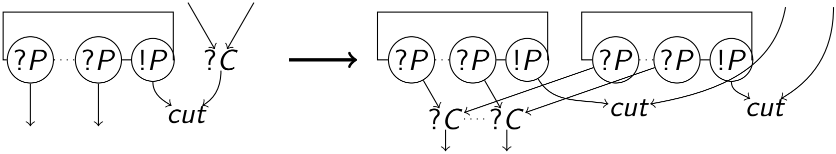
Proof net $\simeq$ syntactic tree + information on duplication

Cut elimination

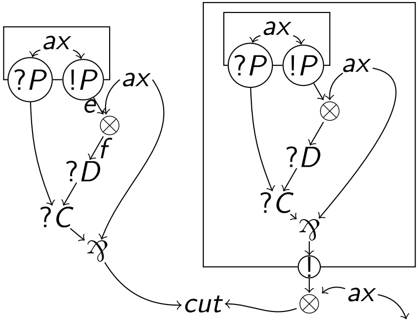


Cut elimination





Stratification : study of $(\lambda x.xx)(\lambda x.xx.)$



Reason for non-normalization?

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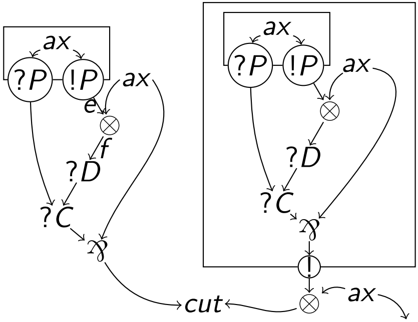

Reason for non-normalization?

- Self application?

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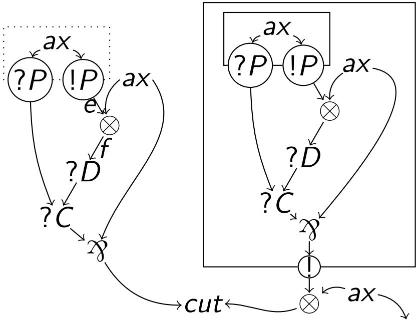

Reason for non-normalization?

- Self application? ✗ The complexity disappears with the box.

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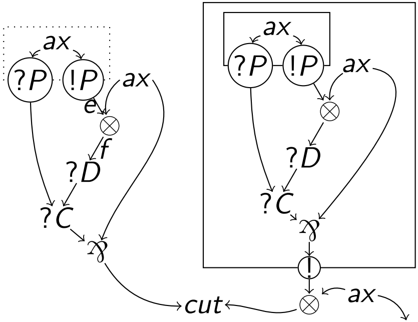

Reason for non-normalization?

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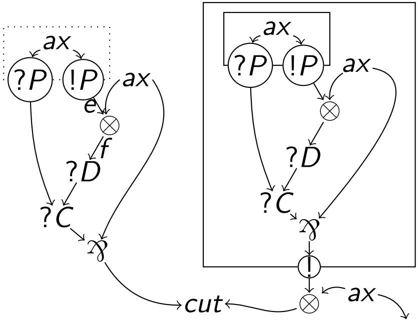
- Self duplication?

Stratification : study of $(\lambda x.xx)(\lambda x.xx.)$



Reason for non-normalization?

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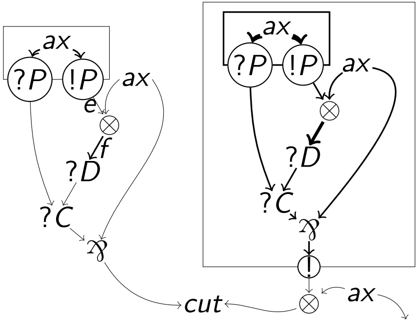

Reason for non-normalization?

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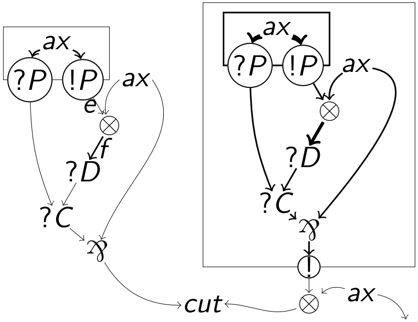
- Self duplication? ∼ Too vague to be a criterium.
- Problem of strata?

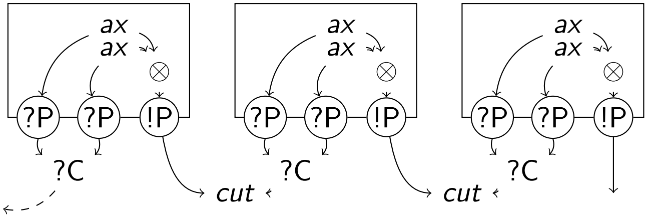
Stratification : study of $(\lambda x.xx)(\lambda x.xx.)$



Reason for non-normalization?

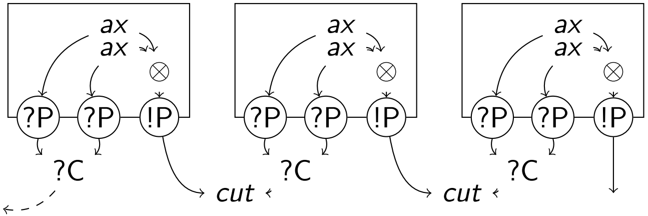
- Self application? ✗ The complexity disappears with the box.
- Self duplication? $\sim$ Too vague to be a criterium.
- Problem of strata? ✓ Stratified $\Rightarrow$ elementary time

Dependence control



Reduces in exponential time

Dependence control



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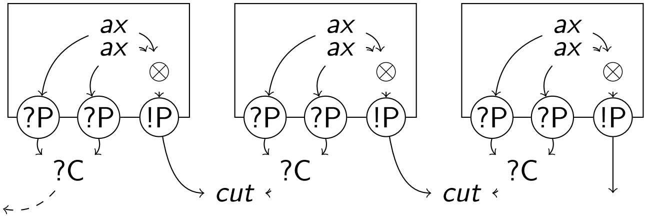
Solution

At most one auxiliary door by box

Dependence control

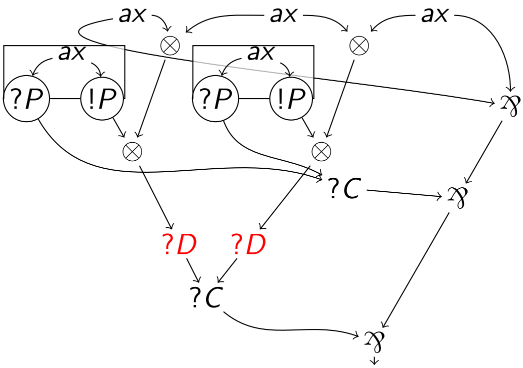


Reduces in exponential time

Solution

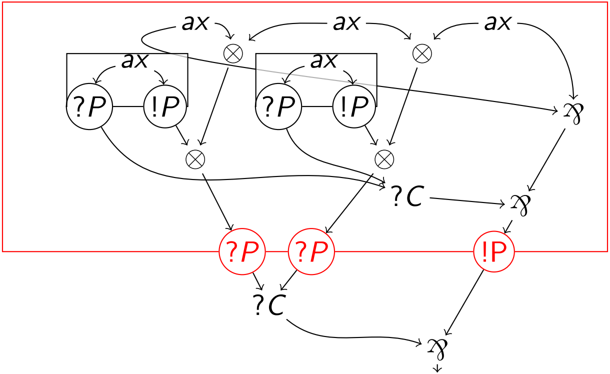
At most one auxiliary door by box + stratification ⇒ Poly time

LLL (Girard 1995)



Stratification : Neither digging, nor dereliction ($?D$ links).

LLL (Girard 1995)



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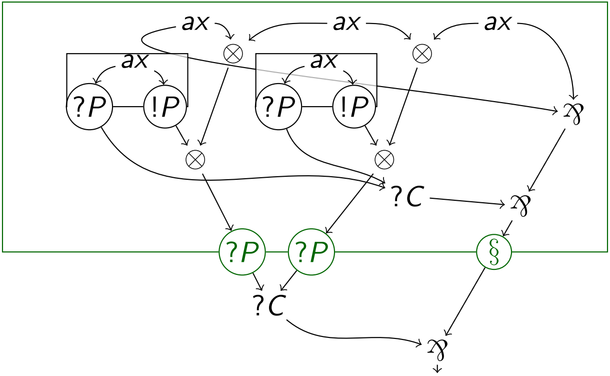

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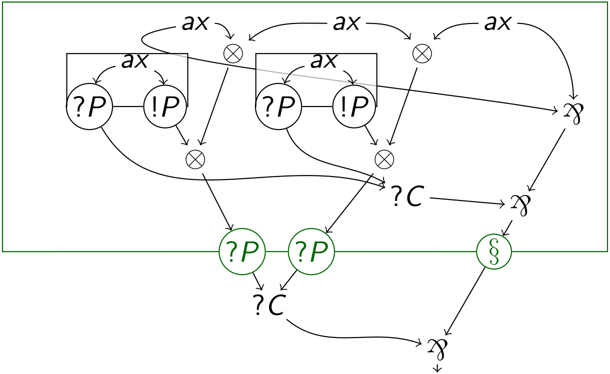
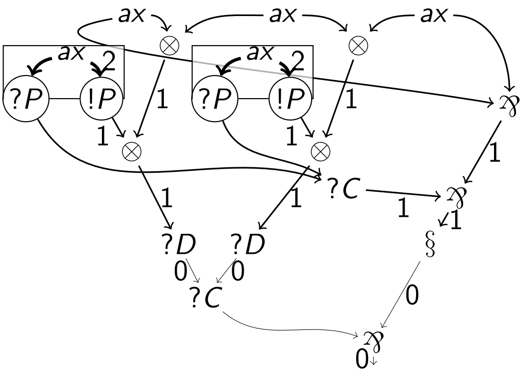


Stratification : Neither digging, nor dereliction ($?D$ links)

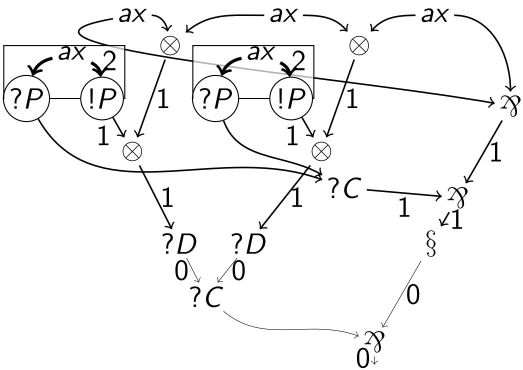
Boxes = duplication, no relation a priori with stratification.

L^4 (Baillot and Mazza 2010)



Explicit levels on edges.

L^4 (Baillot and Mazza 2010)



Explicit levels on edges.

$$\S A \multimap \S B \equiv \S(A \multimap B)$$

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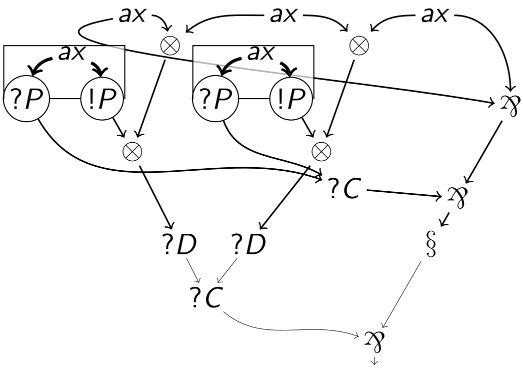

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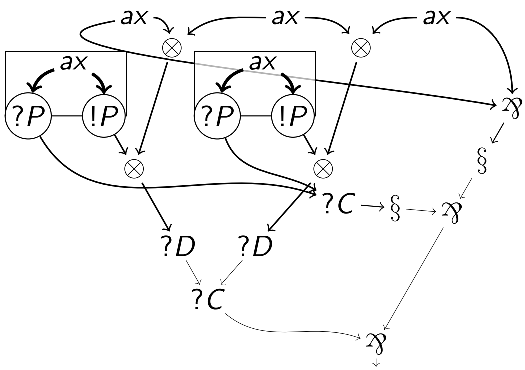

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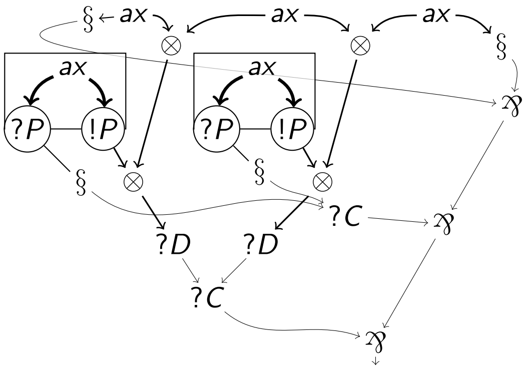

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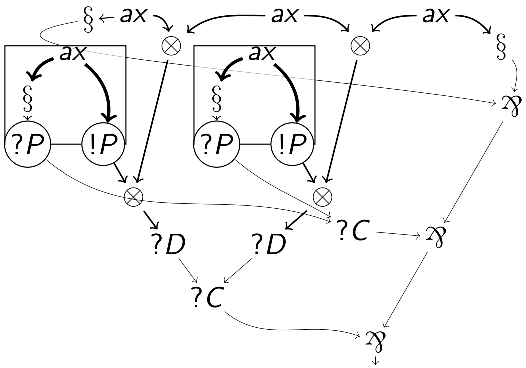

Explicit levels on edges.

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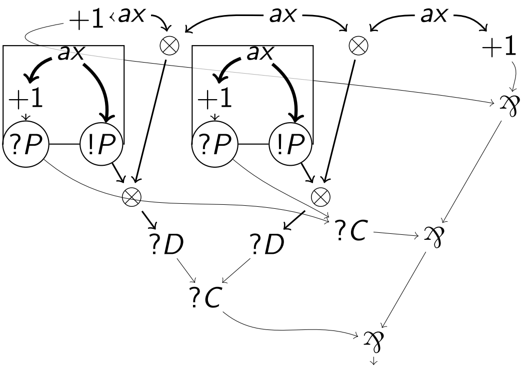
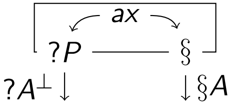
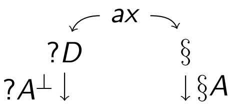
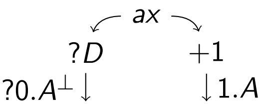
L_0^4 (Baillot and Mazza 2010)



Stratification : The § are moved up.

$$§(§X \multimap Y) \rightsquigarrow (2.X \multimap 1.Y)$$

Light linear logic systems capturing P

	Logic systems
	LLL Weak bound ✓ Strong bound ✓
	L^4 Weak bound ✓ Strong bound ?
	L^4_0 Weak bound ? Strong bound ?

Light linear logic systems capturing P

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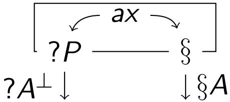
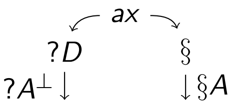
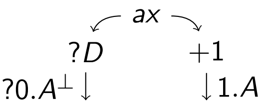
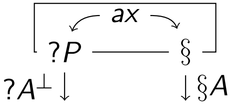
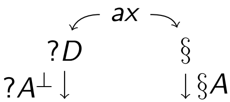
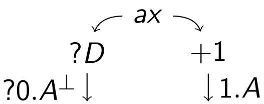
	LLL Weak bound ✓ Strong bound ✓
	L^4 Weak bound ✓ Strong bound ?
	L^4_0 Weak bound ✓ Strong bound ?

Light linear logic systems capturing P

	Logic systems	Type systems
	LLL Weak bound ✓ Strong bound ✓	$DLAL$ Weak bound ✓ Strong bound ✓
	L^4 Weak bound ✓ Strong bound ?	
	L^4_0 Weak bound ✓ Strong bound ?	

Light linear logic systems capturing P

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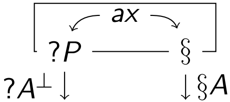
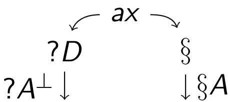
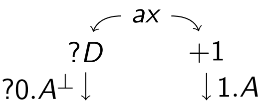
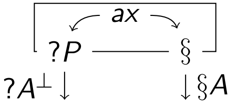
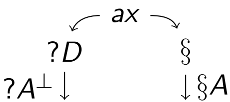
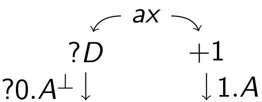
	LLL Weak bound ✓ Strong bound ✓	$DLAL$ Weak bound ✓ Strong bound ✓
	L^4 Weak bound ✓ Strong bound ?	
	L^4_0 Weak bound ✓ Strong bound ?	$DLALL_0$

Light linear logic systems capturing P

	Logic systems	Type systems
	LLL Weak bound ✓ Strong bound ✓	$DLAL$ Weak bound ✓ Strong bound ✓
	L^4 Weak bound ✓ Strong bound ?	
	L^4_0 Weak bound ✓ Strong bound ?	$DLALL_0$ Weak bound ? Strong bound ?

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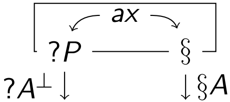
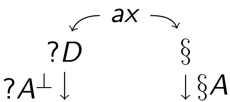
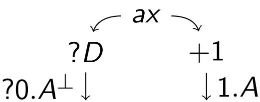
	LLL Weak bound ✓ Strong bound ✓	$DLAL$ Weak bound ✓ Strong bound ✓
	L^4 Weak bound ✓ Strong bound ?	
	L^4_0 Weak bound ✓ Strong bound ?	$DLALL_0$ Weak bound ? Strong bound ?

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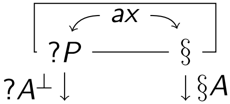
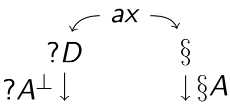
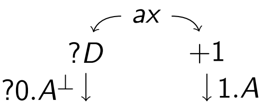
	LLL Weak bound ✓ Strong bound ✓	$DLAL$ Weak bound ✓ Strong bound ✓
	L^4 Weak bound ✓ Strong bound ?	
	L^4_0 Weak bound ✓ Strong bound ?	$DLALL_0$ Weak bound ? Strong bound ?

Proving a strong bound

Technique to prove a bound

Bound on reduction = quantity decreasing for each reduction

- Linear bound for MALL : size of the net
- Depth by depth strategy for LLL : size at depth i

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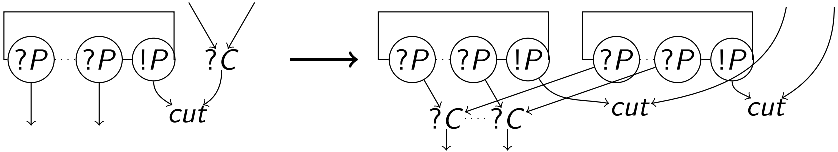

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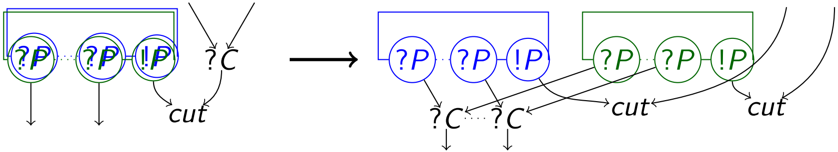
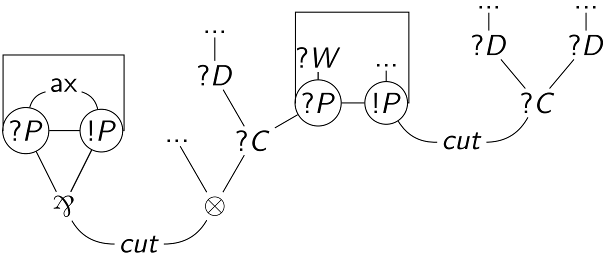


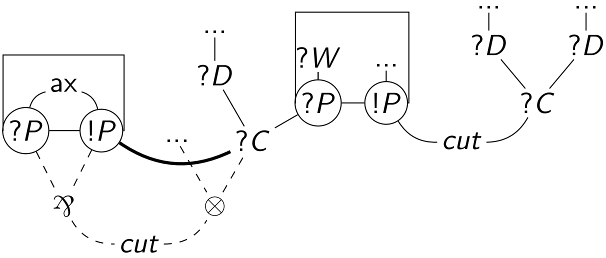
Solution

Anticipate.Consider all the possible duplicates of each link.

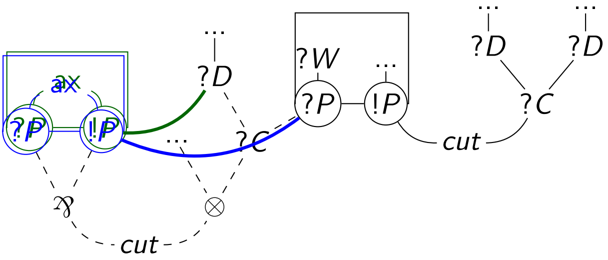
Intuitive notion of duplicates



Intuitive notion of duplicates

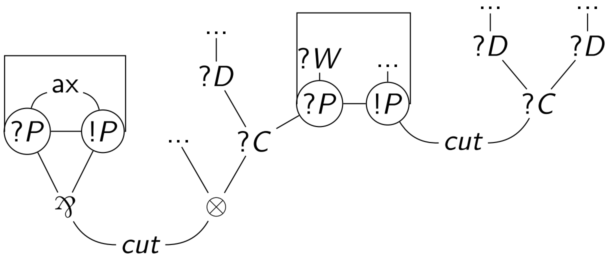


Intuitive notion of duplicates

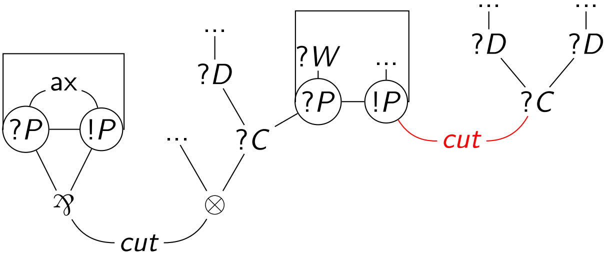


Duplicates of B : $\{1, 2\}$?

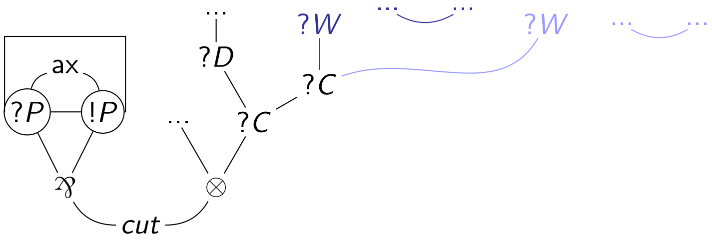
Intuitive notion of duplicates



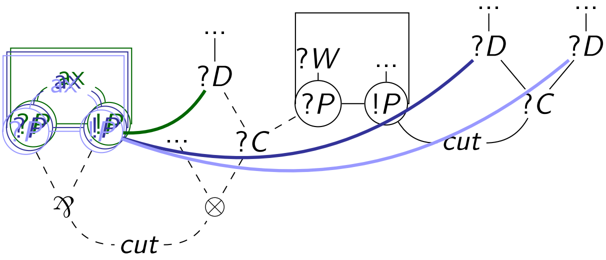
Intuitive notion of duplicates



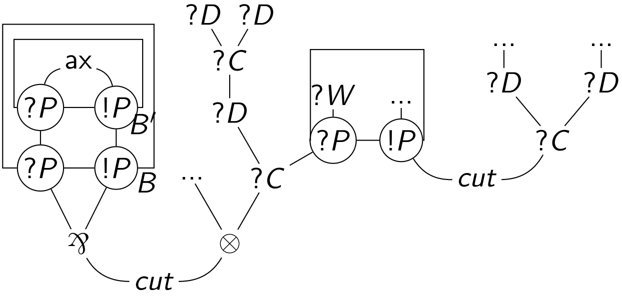
Intuitive notion of duplicates



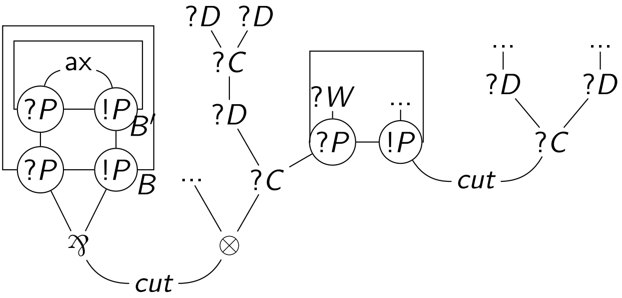
Intuitive notion of duplicates



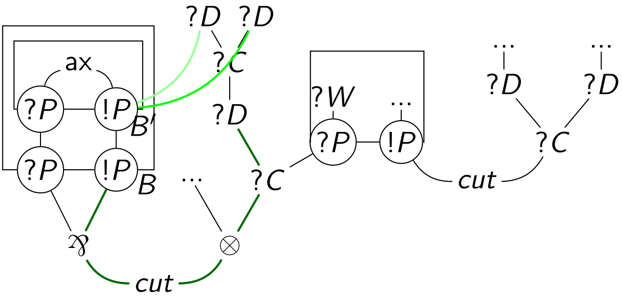
Duplicates of B : $\{1, 2.1, 2.2\}$



Duplicates of B' ?

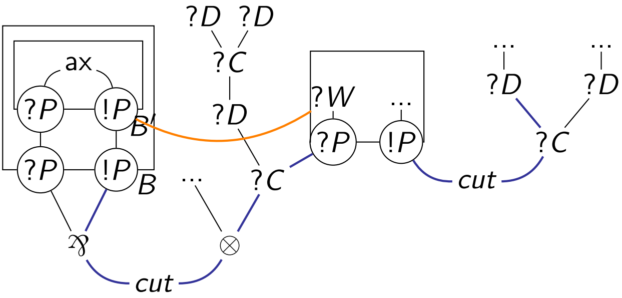


Duplicates of B' ? Depends on the duplicate of B we consider



- Duplicates of B', in the duplicate 1 of $B = \{1, 2\}$

Duplicates of B' ? Depends on the duplicate of B we consider



- Duplicates of B', in the duplicate 1 of $B = \{1, 2\}$
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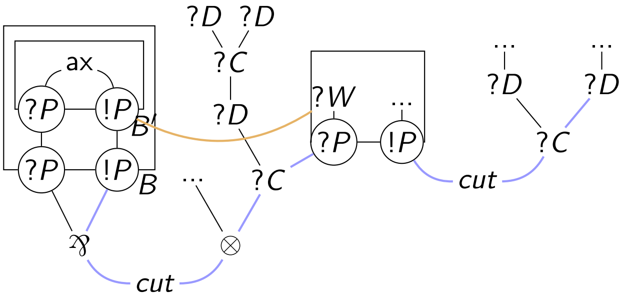

- Duplicates of B', in the duplicate 1 of $B = \{1, 2\}$
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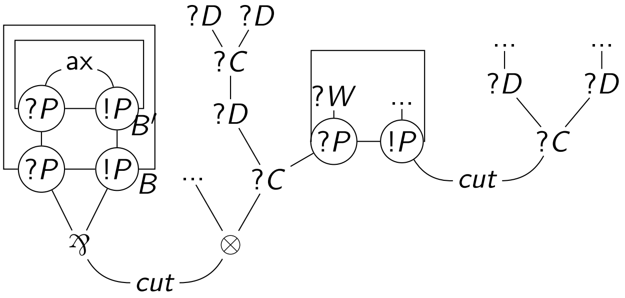

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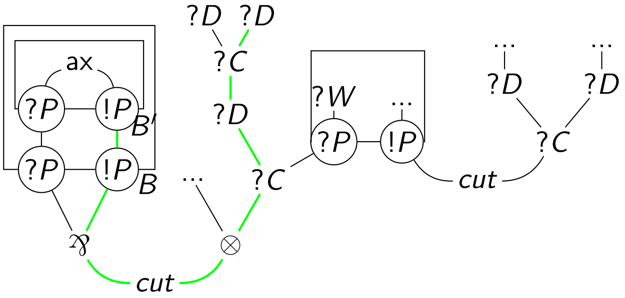
Duplicates of B' : $\{\varepsilon \circ 2.1, \varepsilon \circ 2.2, 1 \circ 1, 2 \circ 1\}$



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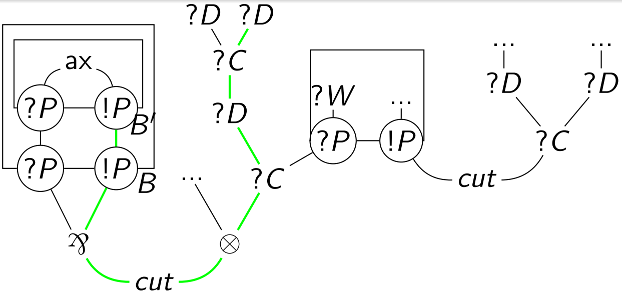
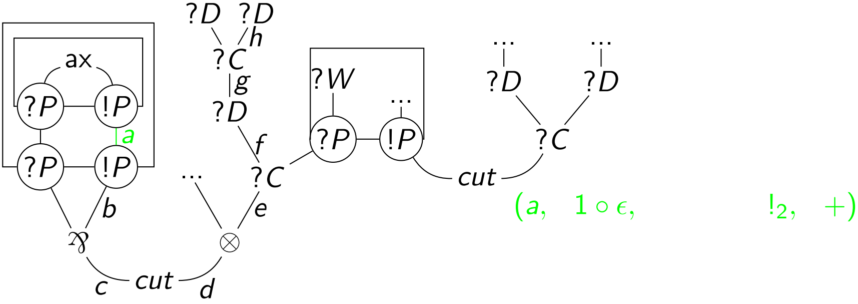
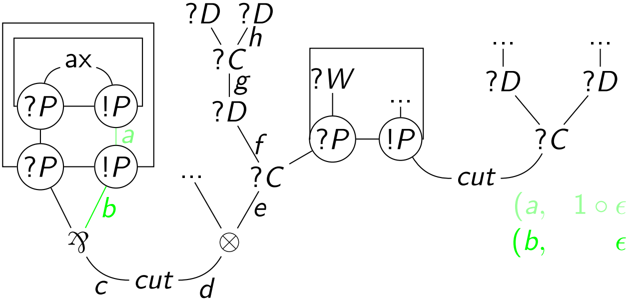


Context

$$C_G = E_G \times \mathcal{P} \times \mathcal{T} \times \{+,-\}$$

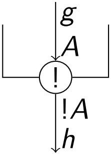
- $e \in E_G$ the edge we are on
- $U \in \mathcal{P}$ potential for e : duplicate for every box containing e
- $V \in \mathcal{T}$ trace : history of the path
- $s \in \{+,-\}$: direction we are going


ax
?P
!P
?P
!P
a
b
⅋
c
cut
d
⊗
...
?D
?D
h
?C
g
?D
f
?C
e
?W
?P
!P
...
cut
...
...
?D
?D
?C
(a, 1 ∘ ϵ, !₂, +)

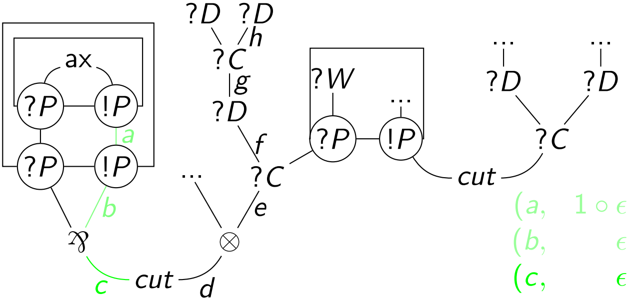


$(a,\quad 1 \circ \epsilon,\quad !_2,\quad +)$

$(b,\quad \epsilon,\quad !_1 :: !_2,\quad +)$



$$(g, t \circ U, V, +) \rightsquigarrow (h, U, !_t :: V, +)$$



$$(a, \quad 1 \circ \epsilon, \quad !_2, \quad +)$$
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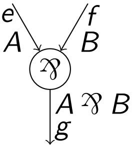
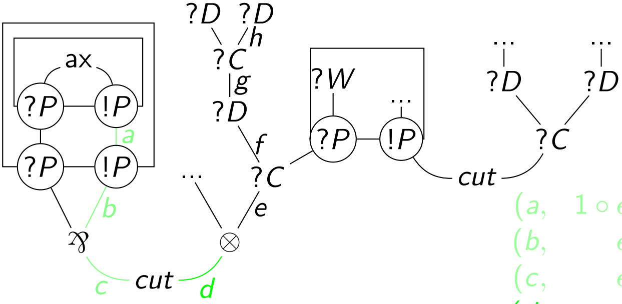
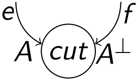


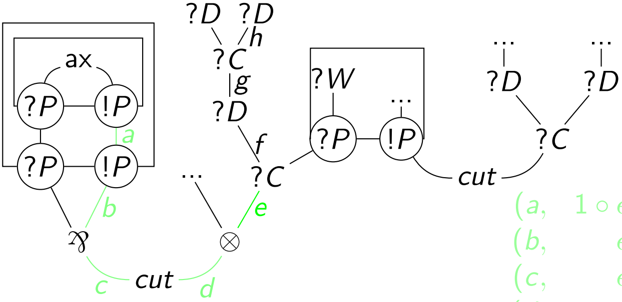
$$(e, U, V, +) \rightsquigarrow (g, U, \parr_l :: V, +)$$
$$(f, U, V, +) \rightsquigarrow (g, U, \parr_r :: V, +)$$



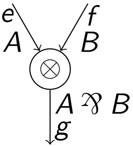
$$
\begin{array}{llll}
(a, & 1 \circ \epsilon, & !_2, & +) \\
(b, & \epsilon, & !_1 :: !_2, & +) \\
(c, & \epsilon, & ⅋_r :: !_1 :: !_2, & +) \\
(d, & \epsilon, & ⅋_r :: !_1 :: !_2, & -)
\end{array}
$$



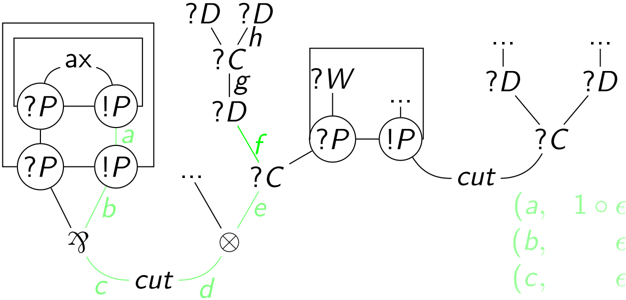
$$(e, U, V, +) \rightsquigarrow (f, U, V, -)$$



$(a,$	$1 \circ \epsilon,$	$!_2,$	$+)$
$(b,$	$\epsilon,$	$!_1 :: !_2,$	$+)$
$(c,$	$\epsilon,$	$⅋_r :: !_1 :: !_2,$	$+)$
$(d,$	$\epsilon,$	$⅋_r :: !_1 :: !_2,$	$-)$
$(e,$	$\epsilon,$	$!_1 :: !_2,$	$-)$



$$(g, U, ⅋_l :: V, -) \rightsquigarrow (e, U, V, -)$$
$$(g, U, ⅋_r :: V, -) \rightsquigarrow (f, U, V, -)$$



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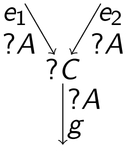
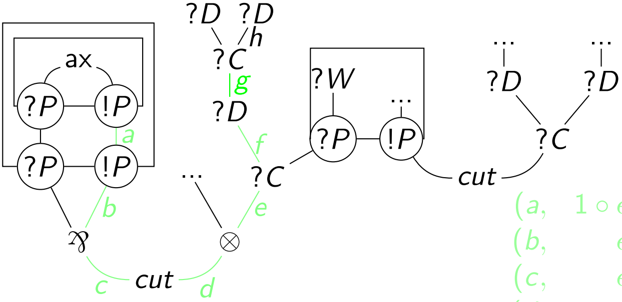
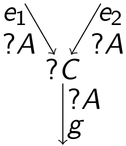
$(d,$	$\epsilon,$	$⅋_r :: !_1 :: !_2,$	$-)$
$(e,$	$\epsilon,$	$!_1 :: !_2,$	$-)$
$(f,$	$\epsilon,$	$!_\varepsilon :: !_2,$	$-)$



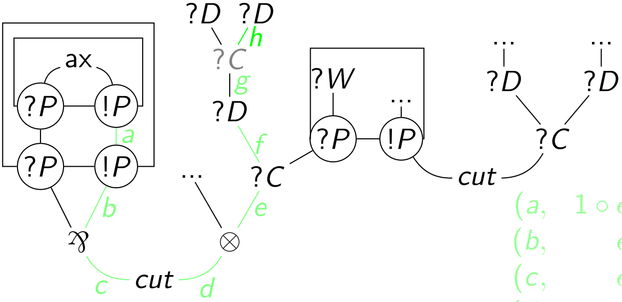
$$(e_i, U, ?_t :: V, +) \rightsquigarrow (g, U, ?_{i.t} :: V, +)$$



$$(a,\quad 1\circ\epsilon,\quad !_2,\quad +)$$
$$(b,\quad \epsilon,\quad !_1 :: !_2,\quad +)$$
$$(c,\quad \epsilon,\quad ⅋_r :: !_1 :: !_2,\quad +)$$
$$(d,\quad \epsilon,\quad ⅋_r :: !_1 :: !_2,\quad -)$$
$$(e,\quad \epsilon,\quad !_1 :: !_2,\quad -)$$
$$(f,\quad \epsilon,\quad !_\varepsilon :: !_2,\quad -)$$
$$(g,\quad \epsilon,\quad !_2,\quad -)$$



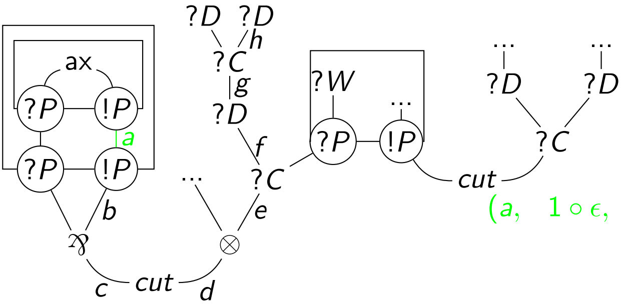
$$(e_i, U, ?_t :: V, +) \rightsquigarrow (g, U, ?_{i.t} :: V, +)$$



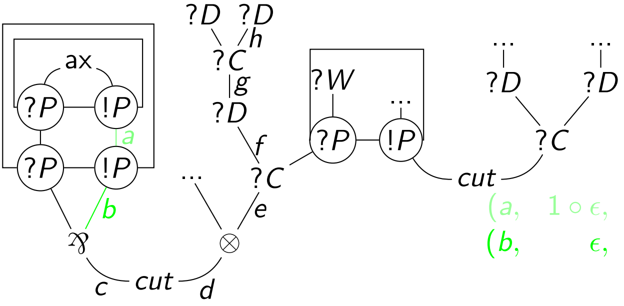
2 is a duplicate of B' in $(B, 1)$

$2 \circ 1$ is a duplicate of B'

$(a,$	$1 \circ \epsilon,$	$!_2,$	$+)$
$(b,$	$\epsilon,$	$!_1 :: !_2,$	$+)$
$(c,$	$\epsilon,$	$⅋_r :: !_1 :: !_2,$	$+)$
$(d,$	$\epsilon,$	$⅋_r :: !_1 :: !_2,$	$-)$
$(e,$	$\epsilon,$	$!_1 :: !_2,$	$-)$
$(f,$	$\epsilon,$	$!_\varepsilon :: !_2,$	$-)$
$(g,$	$\epsilon,$	$!_2,$	$-)$
$(h,$	$\epsilon,$	$!_\varepsilon,$	$-)$

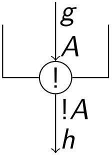


$(a, \quad 1 \circ \epsilon, \qquad !_{2.1.1.2}, \quad +)$

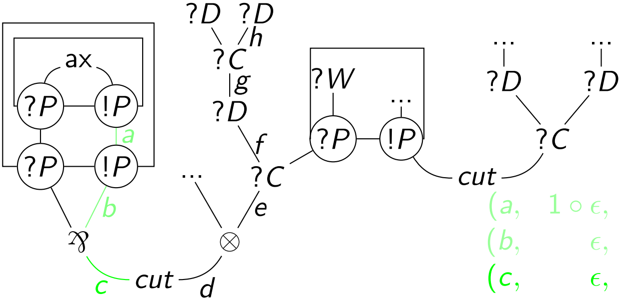


$(a, \quad 1 \circ \epsilon, \quad !_{2.1.1.2}, \quad +)$

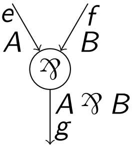
$(b, \quad \epsilon, \quad !_1 :: !_{2.1.1.2}, \quad +)$



$$(g, t \circ U, V, +) \rightsquigarrow (h, U, !_t :: V, +)$$



$$(a,\quad 1 \circ \epsilon,\quad !_{2.1.1.2},\quad +)$$
$$(b,\quad \epsilon,\quad !_1 :: !_{2.1.1.2},\quad +)$$
$$(c,\quad \epsilon,\quad \parr_r :: !_1 :: !_{2.1.1.2},\quad +)$$



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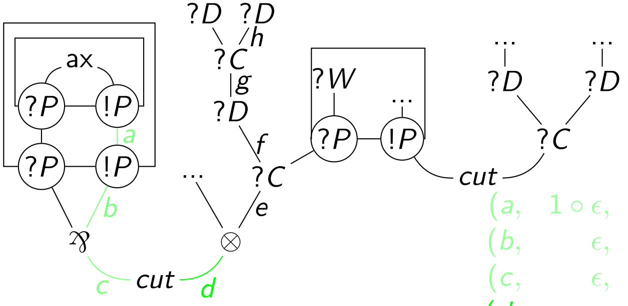
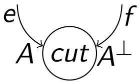
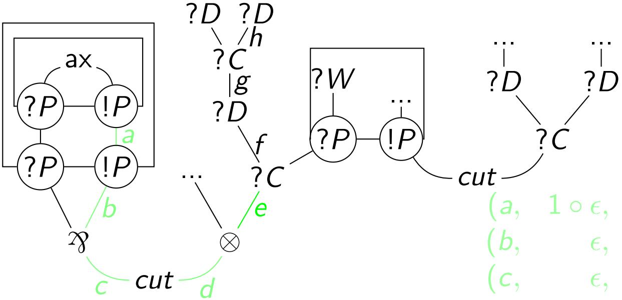


$$\begin{array}{lllll}
(a, & 1\circ\epsilon, & & !_{2.1.1.2}, & +)\\
(b, & \epsilon, & !_1 :: & !_{2.1.1.2}, & +)\\
(c, & \epsilon, & \parr_r :: !_1 :: & !_{2.1.1.2}, & +)\\
(d, & \epsilon, & \parr_r :: !_1 :: & !_{2.1.1.2}, & -)
\end{array}$$



$$(e, U, V, +) \rightsquigarrow (f, U, V, -)$$



$(a, \quad 1 \circ \epsilon, \quad !_{2.1.1.2}, \quad +)$
$(b, \quad \epsilon, \quad !_1 :: !_{2.1.1.2}, \quad +)$
$(c, \quad \epsilon, \quad \parr_r :: !_1 :: !_{2.1.1.2}, \quad +)$
$(d, \quad \epsilon, \quad \parr_r :: !_1 :: !_{2.1.1.2}, \quad -)$
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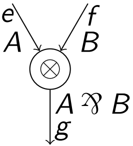

$$(g, U, \parr_l :: V, -) \rightsquigarrow (e, U, V, -)$$
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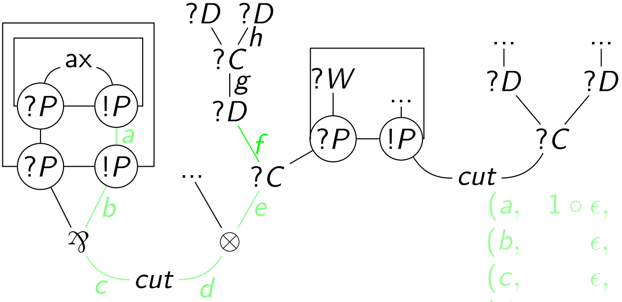
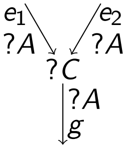
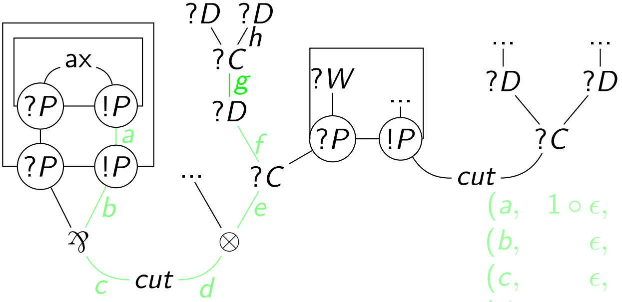


$(a,$	$1 \circ \epsilon,$	$!_{2.1.1.2},$	$+)$
$(b,$	$\epsilon,$	$!_1 :: !_{2.1.1.2},$	$+)$
$(c,$	$\epsilon,$	$\parr_r :: !_1 :: !_{2.1.1.2},$	$+)$
$(d,$	$\epsilon,$	$\parr_r :: !_1 :: !_{2.1.1.2},$	$-)$
$(e,$	$\epsilon,$	$!_1 :: !_{2.1.1.2},$	$-)$
$(f,$	$\epsilon,$	$!_\epsilon :: !_{2.1.1.2},$	$-)$



$$(e_i, U, ?_t :: V, +) \rightsquigarrow (g, U, ?_{i.t} :: V, +)$$



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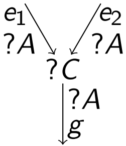

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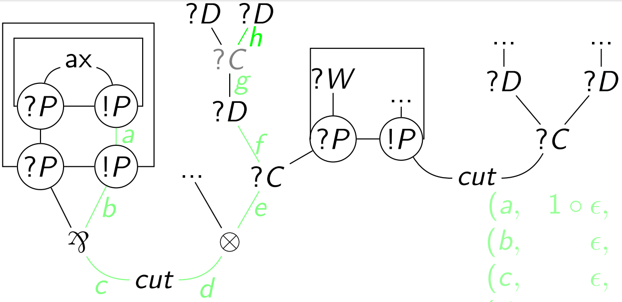

2.1.1.2 is a duplicate of B' in $(B,1)$?

$2.1.1.2 \circ 1$ is a duplicate of B' ?

$(a,$	$1 \circ \epsilon,$	$!_{2.1.1.2},$	$+)$
$(b,$	$\epsilon,$	$!_1 :: !_{2.1.1.2},$	$+)$
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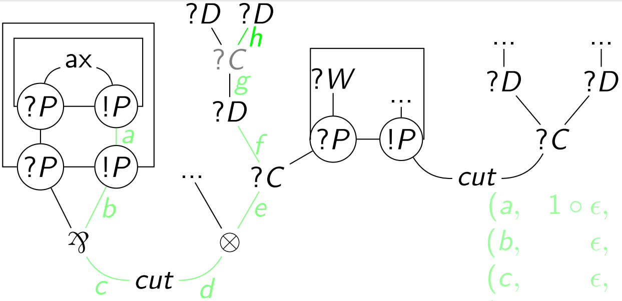

$(a,$	$1 \circ \epsilon,$	$!_{2.1.1.2},$	$+)$
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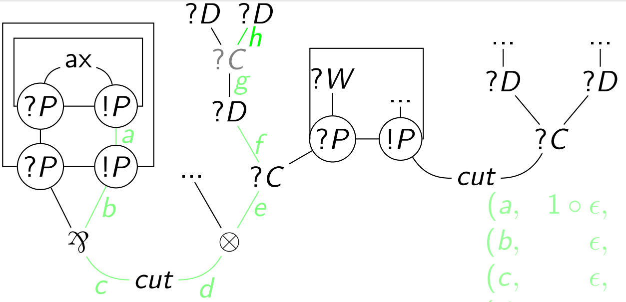

$(a, \quad 1\circ\epsilon, \quad !_{2.1.1.2}, \quad +)$
$(b, \quad \epsilon, \quad !_1 :: !_{2.1.1.2}, \quad +)$
$(c, \quad \epsilon, \quad ⅋_r :: !_1 :: !_{2.1.1.2}, \quad +)$
$(d, \quad \epsilon, \quad ⅋_r :: !_1 :: !_{2.1.1.2}, \quad -)$
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~~2.1.1.2 is a duplicate of B' in $(B, 1)$?~~

~~$2.1.1.2 \circ 1$ is a duplicate of B' ?~~

copy

$t \in R_G(B, U)$ if $(principal(B), U, !_t, +) \mapsto^* (e, W, !_\epsilon, +)$

Dal Lago 2006

copy

$t \in R_G(B, U)$ if $(principal(B), U, !_t, +) \mapsto^* (e, W, !_\epsilon, +)$

U canonical potential of B

$U \in L_G$ if composed of a copy for each box containing B.

Dal Lago 2006

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U* canonical potential of *B

$U \in L_G$ if composed of a copy for each box containing B.

Dal Lago's theorem

$W_G \leq$ max reduction length $\leq T_G \leq poly(max_{B,U}(|R_G(B, U)|))$

Strong bound for *LLL* (Dal Lago 2006)

$$|R_G(B,U)| = |\{t/(princ(B),U,!_t,+) \mapsto^* (e,W,!_\epsilon,\_)\}|$$

Strong bound for *LLL* (Dal Lago 2006)

Dependence control

The correspondance $t \mapsto (e, W)$ is injective

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$$
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|R_G(B,U)| &= |\{t/(princ(B), U, !_t, +) \mapsto^* (e, W, !_\epsilon, \_)\}| \\
&\leq |\{(e, W)/e \in G\}| \\
&\leq |\{(e, W)/e \in G, |W| = |U|\}| \\
&\leq |G| * max_{\partial(e)=\partial(B)} |L_G(e)|
\end{aligned}
$$

$$
max_{\partial(B)=d} |R_G(B,U)| \leq |G| * max_{\partial(e)=d} |L_G(e)|
$$

Strong bound for LLL (Dal Lago 2006)

Dependence control

The correspondance $t \mapsto (e, W)$ is injective

Stratification

If $(e, U, V, \_) \mapsto (f, U', V', \_)$, Then $|U| + |V|_{!,?,\S} = |U'| + |V'|_{!,?,\S}$.

$$
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\end{aligned}
$$

$$max_{\partial(B)=d} |R_G(B,U)| \leq |G| * max_{\partial(e)=d} |L_G(e)|$$

$$max_{\partial(B)=d} |R_G(B,U)| \leq |G| * (max_{\partial(B)<d} |R_G(B,U)|)^d$$

First try to adapt the *LLL* proof

$$|R_G(B,U)| = |\{t/(princ(B),U,!_t,+) \mapsto^* (e,W,!_\epsilon,\_)\}|$$

First try to adapt the LLL proof

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\end{aligned}
$$

First try to adapt the LLL proof

Dependence control

The correspondance $t \mapsto (e, W)$ is injective

Stratification

If $(e, U, V, \_) \mapsto (f, U', V', \_)$, do we have
$I(e) + |V|_{!,?,\S} = I(f) + |V'|_{!,?,\S}$?

$$
\begin{aligned}
|R_G(B,U)| &= |\{t/(princ(B), U, !_t, +) \mapsto^* (e, W, !_\epsilon, \_)\}| \\
&\leq |\{(e, W)/e \in G\}|
\end{aligned}
$$

First try to adapt the LLL proof

Dependence control

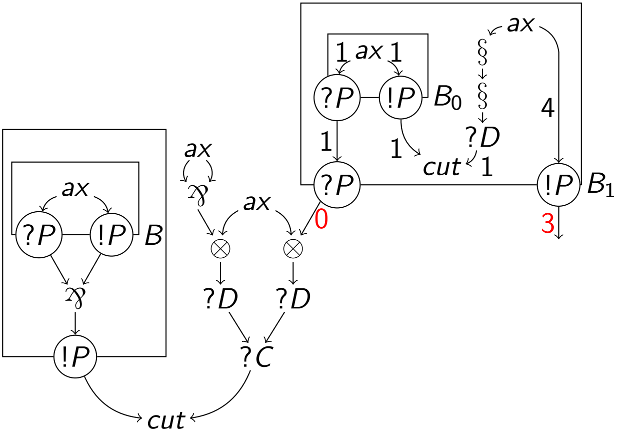
The correspondance $t \mapsto (e, W)$ is injective

Stratification

If $(e, U, V, \_) \mapsto (f, U', V', \_)$, ~~$l(e) + |V|_{!,?,\S} = l(f) + |V'|_{!,?,\S}$.~~

$$|R_G(B, U)| = |\{t/(princ(B), U, !_t, +) \mapsto^* (e, W, !_\epsilon, \_)\}|$$
$$\leq |\{(e, W)/e \in G\}|$$

Doors of a same box with different levels



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Dependence control

The correspondance $t \mapsto (e, W)$ is injective

Stratification

If $(e, U, V, \_) \mapsto (f, U', V', \_)$, ~~$l(e) + |V|_{!,?,\S} = l(f) + |V'|_{!,?,\S}$.~~

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If $(e, U, V, \_) \mapsto (f, U', V', \_)$, ~~$l(e) + |V|_{!,?,§} = l(f) + |V'|_{!,?,§}$~~.

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Dependence control

The correspondance $t \mapsto (e, W)$ is injective

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If $(e, U, V, \_) \mapsto (f, U', V', \_)$, $\cancel{I(e) + |V|_{!,?,\S} = I(f) + |V'|_{!,?,\S}}$.

$$\begin{aligned} |R_G(B,U)| &= |\{t/(princ(B), U, !_t, +) \mapsto^* (e, W, !_\epsilon, \_)\}| \\ &\leq |\{(e,W)/e \in G\}| \\ &\leq |\{(e,W)/e \in G, I(e) = I(B)\}| \\ &\leq |G| * max_{I(e)=I(B)} |L_G(e)| \\ max_{I(B)=d} |R_G(B,U)| &\leq |G| * max_{I(e)=d} |L_G(e)| \end{aligned}$$

First try to adapt the LLL proof

Dependence control

The correspondance $t \mapsto (e, W)$ is injective

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If $(e, U, V, \_) \mapsto (f, U', V', \_)$, ~~$l(e) + |V|_{!,?,\S} = l(f) + |V'|_{!,?,\S}$.~~

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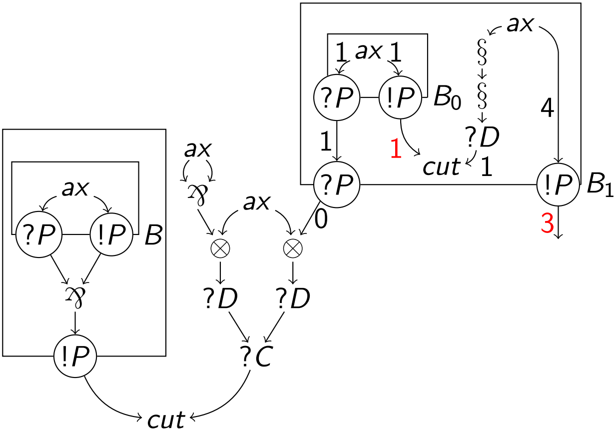
$$\leq |\{(e,W)/e \in G, l(e) = l(B)\}|$$
$$\leq |G| * max_{l(e)=l(B)} |L_G(e)|$$
$$max_{l(B)=d} |R_G(B,U)| \leq |G| * max_{l(e)=d} |L_G(e)|$$

impossible to conclude

Box of low level inside box of high level



Idea of the solution

Induction on $|U|$ $\sim$ depth by depth interaction $\sim$ strategy by depth
Strategy by levels $\sim$ restricted potentials $|U^{/l}|$

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$t \in R_G^{/l}(B, U)$ iff $(principal(B), U, !_t, +) \mapsto^*_{/l} C_F$

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Canonical potential restricted at level l

$$L_G^{/l}(B) = \begin{cases} \{\varepsilon\} & \text{if } B \text{ has level } 0 \\ \bigcup_{U \in L_G^{/l}(B')} R_G^{/l-1}(B,U) \circ U & \text{if } B \subset B' \end{cases}$$

Strong bound L^4

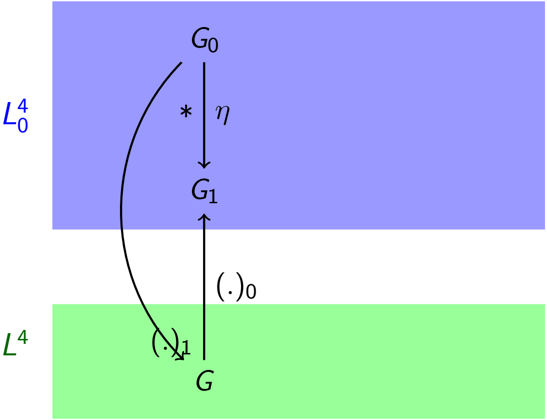
Dependence control

If $t \in R_G^{/l}(B, U)$, then $(principal(B), U, !_\epsilon, +) \rightsquigarrow^* (e, W, !_\epsilon, \_)$
The mapping $t \mapsto (e, W^{/l})$ is injective

Bound L^4

If $(e, U, V, \_) \rightsquigarrow (f, U', V', \_)$, Then $l(e) + |V|_{!,?,\S} = l(f) + |V'|_{!,?,\S}$.

$$\begin{aligned} |R_G^{/l}(B, U)| &\leq |\{(e, W)/W \in L_G^{/l}(e)\}| \\ &\leq |G| * max|L_G^{/l}(e)| \\ max|R_G^{/l}(B, U)| &\leq |G| * max|L_G^{/l}(e)| \\ max|R_G^{/l}(B, U)| &\leq |G| * (max|R_G^{/l-1}(B, U)|)^{\partial(G)} \end{aligned}$$



$$T_{G_0} \leq T_{G_1} \leq T_G \leq poly(W_G) \leq poly(|G|) \leq poly(|G_0|)$$

$DLALL_0$

Grammar of types of $DLALL_0$

$$A, B ::= n.X \mid A \multimap B \mid A \Rightarrow B \mid \forall X, A$$

$$x_i : {A_i}^{m_i}\,\boldsymbol{;}\ y_j : {B_j}^{n_j} \vdash t : A^n$$

$$\frac{}{\boldsymbol{;}\ x : A^{n+z} \vdash x : z.A^n}\ ax$$

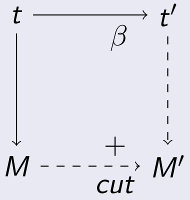
$$\frac{\Gamma_1\boldsymbol{;}\ \Delta_1 \vdash t : A \Rightarrow B^n \qquad \boldsymbol{;}\ z : C^{m+1} \vdash u : A^{n+1}}{\Gamma_1, z : C^m\boldsymbol{;}\ \Delta_1 \vdash tu : B^n}\ \Rightarrow_e$$

$$\frac{\Gamma, x : A^n\boldsymbol{;}\Delta \vdash t : B^n}{\Gamma\boldsymbol{;}\Delta \vdash \lambda x.t : A \Rightarrow B^n}\ \Rightarrow_i$$

Stroung bound for $DLALL_0$

If t is typable in $DLALL_0$, we assign an L^4_0 intuitionnistic net to it.

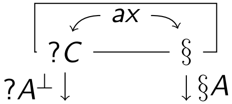
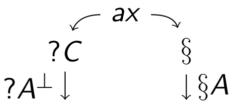
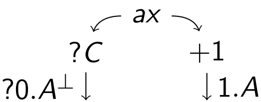
Simulation theorem



Bound $DLALL_0$

If t is typable in $DLALL_0$ with level $\leq l$, then t strongly normalizes in $\leq p_l(|t|)$ steps.

Light linear logic systems capturing P

	Logic systems	Type systems
	*LLL* Weak bound ✓ Strong bound ✓	*DLAL* Weak bound ✓ Strong bound ✓
	L^4 Weak bound ✓ Strong bound ✓	
	L^4_0 Weak bound ✓ Strong bound ✓	$DLALL_0$ Weak bound ✓ Strong bound ✓

Future works

- Prove a strong bound for L^{3a}.

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- Find a characterization of stratification and dependence control

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- Prove a strong bound for L^{3a}.
- Find a characterization of stratification and dependence control
- Generalize context semantics to add probabilities or other constructions