Multiform preorders and partial combinatory algebras

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Motivation

Context

- Aim: get a more conceptual understanding of realizability toposes
- Main tool: fibred posets $\mathcal{A} : |\mathcal{A}| \to \textbf{Set}$ (e.g. triposes)

Approach – Philosophy

- **Fibred posets are a great tool to organize proof/realizability interpretations**
- Fibred posets can (and should) be viewed as generalized posets
- For ordinary posets, we have a great theory of (pre)sheaf toposes
- Generalize as much as possible from posets to fibred posets
- Example:
  Construction of topos $\textbf{Set}[\mathcal{P}]$ from tripos $\mathcal{P}$ is analogue of sheaftopos $\textbf{Sh}(A)$ from complete Heyting algebra A.
  What is the analogue of the *pre*sheaf topos $\widehat{D}$ for poset D with finite meets?

Realizability over a partial combinatory algebra

- Given a (weak) *partial combinatory algebra (pca)* $\mathcal{A}$, define the fibred poset (tripos)

$$\mathbf{rt}(\mathcal{A}) : |\mathbf{rt}(\mathcal{A})| \to \mathbf{Set}$$

  - predicates on $I \in \mathbf{Set}$: functions $\varphi : I \to P\mathcal{A}$
  - entailment: $\varphi \vdash_I \psi$ iff there exists $e \in \mathcal{A}$ such that $\forall i \, \forall a \in \varphi(i) \, . \, e \cdot a \in \psi(i)$.

- The *realizability topos* $\mathbf{RT}(\mathcal{A}) = \mathbf{Set}[\mathbf{rt}(\mathcal{A})]$ is the category of *partial equivalence relations* and *functional relations* relative to $\mathbf{rt}(\mathcal{A})$.
- Embed $\mathbf{Set}$ into $\mathbf{RT}(\mathcal{A})$ via *regular* functor

$$\Delta : \mathbf{Set} \to \mathbf{RT}(\mathcal{A})$$

- Want to characterize functors

$$\Delta : \mathbf{Set} \to \mathcal{E}$$

  arising this way.
- By Moens' theorem, $(\Delta, \mathbf{RT}(\mathcal{A}))$ is equivalent to a *fibration of toposes*

Moens' theorem

- $\mathbb{C}, \mathbb{D}$ categories with finite limits, $\Delta : \mathbb{C} \to \mathbb{D}$ preserves finite limits
- Glueing construction $$\begin{array}{ccc} \mathrm{Gl}(\Delta) & \longrightarrow & \mathbb{D}{\downarrow}\mathbb{D} \\ {\scriptstyle \mathrm{gl}(\Delta)}\downarrow & \lrcorner & \downarrow{\scriptstyle P_{\mathbb{D}}} \\ \mathbb{C} & \xrightarrow{\Delta} & \mathbb{D} \end{array}$$ gives a fibration $\mathrm{gl}(\Delta)$ with finite limits and extensive internal sums

- $\mathscr{C} : |\mathscr{C}| \to \mathbb{C}$ fibration with finite limits and extensive internal sums
- Define $\Delta : \mathbb{C} \to \mathscr{C}_1$ by $\mathbb{C} \mapsto \sum_X 1$
- Δ preserves finite limites

Theorem (Moens)

These constructions establish an equivalence between finite limit preserving functors with domain $\mathbb{C}$ and lextensive fibrations on $\mathbb{C}$

- See: T Streicher, *Fibred categories à la Jean Bénabou*, 1999-2010

Now let $\mathbb{R}$ be a regular category

- *regular* functors $\Delta : \mathbb{R} \to \mathbb{X}$ into *regular* categories correspond to *prestacks* of regular categories with extensive internal sums
- regular functors into *exact* categories correspond to stacks of exact categories with extensive internal sums

For the last kind of fibration, we introduce a special name

Definition

Let $\mathbb{R}$ be a regular category. A **fibred pretopos** on $\mathbb{R}$ is a stack of exact categories with extensive internal sums.

Part I
The fibred presheaf construction

Fibred presheaf construction

Motivation: non-fibred case

- $\mathbb{C}$ small category with finite limits
- Fibration of sieves: $$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ {\scriptstyle \mathrm{siev}(\mathbb{C})}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{sub}(\widehat{\mathbb{C}})} \\ \mathbb{C} & \xrightarrow{Y} & \widehat{\mathbb{C}} \end{array}$$
- Fibred fibration of sieves: $\Sigma(\bullet) \xrightarrow{\Sigma(\mathrm{siev}(\mathbb{C}))} \Sigma\mathbb{C} \xrightarrow{\mathrm{fam}(\mathbb{C})} \mathbf{Set}$
- $F \in \widehat{\mathbb{C}}$ can be covered by representables: $\sum_i YC_i \overset{e}{\twoheadrightarrow} F$
- Kernel $\sum_{ij} U_{ij} \rightarrowtail \sum_{ij} Y(C_i \times C_j) \cong (\sum_i YC_i) \times (\sum_i YC_i)$ of e is an equivalence relation in $\Sigma(\mathrm{siev}(\mathbb{C}))$ (extensivity!)
- We have $\widehat{\mathbb{C}} \cong \Sigma\mathbb{C}[\Sigma(\mathrm{siev}(\mathbb{C}))]$ (category of (partial) equivalence relations and functional relations)

Fibred presheaf construction

Alternative description of $\Sigma(\mathrm{siev}(\mathbb{C}))$

- Want to express fibred fibration of sieves purely in terms of $\mathrm{fam}(\mathbb{C})$, using 'generating families' instead of real sieves
- $f : (D_j)_{j \in J} \to (C_i)_{i \in I}$ in $\Sigma\mathbb{C}$ can be viewed as "for each $i \in I$, a family of maps into C_i"
- Alternative representation of $\Sigma(\mathrm{siev}(\mathbb{C}))$:
    - predicates on $(C_i)_{i \in I}$: maps $f : (D_j)_{j \in J} \to (C_i)_{i \in I}$, $g : (E_k)_k \to (C_i)_i$
    - $f \vdash g$ iff $\forall j \; \exists k, \; h : D_j \to E_k \, . \, g_k h = f_j$

Fibred presheaf construction

Fibration of sieves on a pre-stack

- $\mathbb{R}$ regular category, $\mathscr{C} : |\mathscr{C}| \to \mathbb{R}$ pre-stack with finite limits
- Define fibred preorder $\mathrm{siev}(\mathscr{C})$ on $|\mathscr{C}|$
  - Predicates on $C \in |\mathscr{C}|$ are maps $f : D \to C$, $g : E \to C$
  - $f \vdash g$ iff there exist h, e, with e cartesian over a regular epimorphism such that
    $$\begin{array}{ccc} \bullet & \xrightarrow{h} & E \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow{f} & C_i \end{array}$$
    commutes

- $\mathsf{siev}(\mathscr{C})$ interprets regular logic
- We can define the category

$$\widetilde{\mathscr{C}} = |\mathscr{C}|[\mathsf{siev}(\mathscr{C})]$$

of equivalence relations and functional relations, and a regular functor

$$\Delta : \mathbb{R} \to \widetilde{\mathscr{C}}, \qquad X \mapsto (1_X, =)$$

- The fibred pretopos $\mathrm{gl}(\Delta)$ is a cocompletion of $\mathscr{C}$, i.e. for a given fibred pretopos $\mathscr{X}$, we have an equivalence

$$\mathbf{Lex}(\mathscr{C}, \mathscr{X}) \simeq \mathbf{Pretop}(\mathrm{gl}(\Delta), \mathscr{X})$$

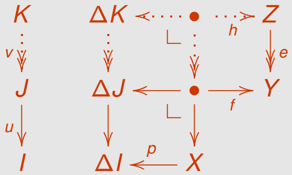
of categories of fibred functors and transformations.

Indecomposables and projectives

Let $\Delta : \mathbb{R} \to \mathbb{X}$ be a regular functor between regular categories

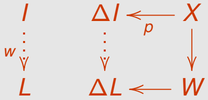
Definition

- Call $p : X \to \Delta I$ **projective**, if for every e, u, and f in the diagram



  there exist v and h making the upper right square commute.

- Call p **indecomposable**, if in the diagram



  there exists a unique w making the square commute

characterization of cocompletions

Theorem

Let $\mathscr{X} : |\mathscr{X}| \to \mathbb{R}$ be a fibred pretopos. $\mathscr{X}$ is a cocompletion in the sense of the previous construction iff

- *the subfibration of $\mathscr{X}$ on indecomposable projectives is closed under finite limits, and*
- *all ojects in $\mathscr{X}$ can be covered by sums of indecomposable projectives.*

Remark: The original pre-stack can be recovered from its cocompletion only up to weak equivalence.

Part II
Multiform preorders

Multiform preorders

Sources, references

- PJW Hofstra, *All realizability is relative*, 2006
- J Longley, *Computability structures, simulations and realizability*, 2011
- N Hoshino, *unpublished work*, 2011

Motivation

- Let $\mathcal{A} : |\mathcal{A}| \to \mathbf{Set}$ be a fibred poset with generic predicate $\mathsf{tr} \in \mathcal{A}_{\mathsf{Prop}}$
- $\mathcal{A}$ is equivalent to $\mathbf{Set}(-, \mathsf{Prop})$, ordered by $f \leq g$ iff $f^*\mathsf{tr} \leq g^*\mathsf{tr}$ in $\mathcal{A}$.
- Consider $f, g : M \to \mathsf{Prop}$, factorize as $M \twoheadrightarrow U \overset{m}{\hookrightarrow} Prop \times Prop$
- Then $f^*\mathsf{tr} \leq g^*\mathsf{tr}$ iff $m_1^*\mathsf{tr} \leq m_2^*\mathsf{tr}$
- $\mathcal{A}$ entirely determined by the set

$$R = \{U \subseteq \mathsf{Prop} \times \mathsf{Prop} \mid \ \pi_l|_U^*\mathsf{tr} \leq \pi_r|_U^*\mathsf{tr}\}$$

- R is downward closed under inclusion, contains the identity relation and is closed under relational composition
- We can do a similar thing for fibred posets with generic *family* of predicates
- This works without choice if we demand the fibred posets to be *pre-stacks*

Multiform preorders

Definition (Longley)

A **multiform preorder** is a triple (I, A, R), where $A = (A_i)_{i \in I}$ is a family of sets, and $R = (R_{ij})_{i,j \in I}$, $R_{ij} \subseteq P(A_i \times A_j)$ is a family of sets of relations, subject to the following axioms.

1. $i, j \in I,\ r \in R_{ij},\ s \subseteq r \implies s \in R_{ij}$
2. $i \in I \implies \mathrm{id} \in R_{ii}$
3. $i, j, k \in I,\ r \in R_{ij},\ s \in R_{jk} \implies sr \in R_{ij}$

Definition

A **monotonous map** between multiform preorders (I, A, R), (J, B, S) is a pair $(u : I \to K, (f_i : A_i \to B_{ui})_{i \in I})$ such that $r \in R_{ij}$ implies $(f_i \times f_j)(r) \in S_{ui,uj}$.

Given $(u, f), (v, g) : (I, A, R) \to (J, B, S)$, we define $(u, f) \leq (v, g)$ iff for all $i \in I$ we have $\{(f_i a, g_i a) \mid a \in A_i\} \in S_{ui,vi}$.
Multiform preorders and monotonous maps form an order-enriched category **UOrd**.

Longley introduced multiform preorders under the name *computability structures*, but he considered different morphisms.

Basic relational objects

Definition

A **basic relational object** (BRO) is a multiform preorder (I, A, R) where I has exactly one element. In this case, we omit the I in the notation and write (A, R).
BROs form an order-enriched category $\mathbf{BRO} \subset \mathbf{UOrd}$.

- Basic relational objects are close to Hofstra's *basic combinatory objects* (BCOs). More precisely, BCOs form a full subcategory of BROs.

The fibred poset associated to a multiform preorder

Let (I, A, R) be a multiform preorder. We define the fibred poset

$$\mathrm{fam}((I, A, R)) : \Sigma(I, A, R) \to \mathbf{Set}$$

as follows.

- a predicate on a set M is a pair $(i \in I, f : M \to A_i)$
- given $(i, f), (j, g) \in \mathrm{fam}((I, A, R))_M$, we define $(i, f) \leq (j, g)$ iff $\{(fm, gm) \mid m \in M\} \in R_{ij}$

Lemma

- **UOrd** *is biequivalent to the the full subcategory of fibred posets on pre-stacks with a family of generic predicates.*
- **BRO** *is biequivalent to the the full subcategory of fibred posets on pre-stacks with generic predicates.*

Closure properties

- **BRO** has small products and an involution operator $(-)^{\mathsf{op}}$
- **UOrd** has small products and coproducts, exponentials and $(-)^{\mathsf{op}}$

Examples

- To any poset $(D, \leq)$ we can associate a BRO $(D, \downarrow\{\leq\})$ ($\downarrow\{\leq\}$ is the set of sub-relations of $\leq$). The associated fibred poset is $\mathbf{Set}(-, D)$ with pointwise order.
- To a partial combinatory algebra (pca) $\mathcal{A}$, we associate the BRO $(\mathcal{A}, R_{\mathcal{A}})$ where $R_{\mathcal{A}}$ consists of all the *subcomputable functions* in $\mathcal{A}$, that is the relations $r \subseteq \mathcal{A} \times \mathcal{A}$ such that there exists $e \in \mathcal{A}$ such that $rab \implies e{\cdot}a = b$.
- In a similar way, a typed pca induces a multiform preorder.

Finite completeness

- Being a 2-category with cartesian products, **UOrd** has an internal notion of (finitely) complete object.
- Call (I, A, R) **finitely complete**, if $\delta : (I, A, R) \to (I, A, R) \times (I, A, R)$ and $! : (I, A, R) \to 1$ have right adjoints.
- Since **UOrd** $\to$ **PFib** is a local equivalence, (I, A, R) is finitely complete iff $\mathrm{fam}((I, A, R))$ has finite meets.
- Concretely, (I, A, R) has binary meets iff there exist

$$\otimes : I \times I \to I$$
$$\wedge : A_i \times A_j \to A_{i \otimes j}$$

  such that for all $i, j \in I$
  - $\{(a \wedge b, a) \mid a \in A_i, b \in A_j\} \in R_{i \otimes j, i}$
  - $\{(a \wedge b, b) \mid a \in A_i, b \in A_j\} \in R_{i \otimes j, j}$
  - $\{(a, a \wedge a) \mid a \in A_i\} \in R_{i, i \otimes i}$

Finite completeness

Definition

We call a multiform preorder (I, A, R) **functional**, for all $i, j \in I$, the elements of R_{ij} are functional relations.

Lemma

If a finitely complete multiform preorder (I, A, R) is functional, then the pairing maps $\wedge : A_i \times A_j \to A_{i \times j}$ are injective.

Example

The BRO $(\mathbb{N}, \mathrm{Prim})$, where Prim is generated by the primitive recursive functions, is finitely complete and functional.
Here, $\wedge$ is given by a primitive recursive coding of pairs.

Existential quantification

Hofstra observed that we can freely adjoin existential quantification to a BCO via a monad D. We can do the same thing for multiform preorders.

Definition

Let (I, A, R) be a multiform preorder, $i, j \in I$, $r \in R_{ij}$. Define $[r] \subseteq P(A_i \times A_j)$ by

$$[r](M, N) :\Leftrightarrow \forall m\, \exists n\, .\, r(m, n)$$

This allows to define a multiform preorder $D(I, A, R) = (I, (PA_i)_{i \in I}, (DR_{ij})_{ij \in I})$, where for $ij \in I$, $R_{ij} = \downarrow \{[r] \mid r \in R_{ij}\}$.

- This gives a lax idempotent monad $D : \mathbf{UOrd} \to \mathbf{UOrd}$.
- D freely adds $\exists$ to a multiform preorder – (I, A, R) has $\exists$ iff it is a D-algebra
- For a pca $\mathcal{A}$, we have $\mathbf{rt}(\mathcal{A}) = D(\mathcal{A}, R_{\mathcal{A}})$
- For a $\wedge$-semi-lattice A, we have $D(A, \downarrow \{\leq\}) \cong (\mathrm{dcl}(A), \downarrow \{\subseteq\})$
- For (I, A, R) with finite meets, we have $\widetilde{(I, A, R)} \simeq \mathbf{Set}[D(I, A, R)]$, in analogy to $\widehat{A} \simeq \mathbf{Sh}(\mathrm{dcl}(A))$ for a meet-semi-lattice A
- By dualizing, we obtain a monad U classifying $\forall$

The monad D_+

- Replacing the powerset P by the non-empty powerset P_+ in the definition of D, we obtain a monad D_+.

Lemma

Longley's category of computability structures is the Kleisli category of **UOrd** *for the monad* D_+.

Lemma

Let $\mathcal{A}, \mathcal{B}$ *be pcas. Then an applicative morphism from* $\mathcal{A}$ *to* $\mathcal{B}$ *is the same thing as a finite meet preserving monotonous map of type*

$$(\mathcal{A}, R_{\mathcal{A}}) \to D_+(\mathcal{B}, R_{\mathcal{B}})$$

- Given a fibred meet-semi-lattice $\mathcal{A} : |\mathcal{A}| \to \mathbf{Set}$, when is its cocompletion $\tilde{\mathcal{A}}$ locally cartesian closed?
- We can answer this question for multiform preorders.

Definition

Let (I, A, R) be a finitely complete multiform preorder. Call (I, A, R) **relationally complete**, if for each pair $j, k \in I$ there exists $j \Rightarrow k \in I$ and $@^j_k \in R_{(j \Rightarrow k) \otimes j, k}$ such that for all $i \in I$ and $r \in R_{i \otimes j, k}$ there exists $\tilde{r} \in R_{i, j \Rightarrow k}$ such that

$$\forall a \in A_i \, \exists h \in A_{j \Rightarrow k} \, . \tilde{r}(a, h) \wedge r(a \wedge -, -) \subseteq @^j_k(h \wedge -, -)$$

Relational completeness

Theorem

Let (I, A, R) be a finitely complete multiform preorder. Then the following are equivalent.

- *(I, A, R) is relationally complete*
- *$D(I, A, R)$ has implication and universal quantification*
- *$(\widetilde{I, A, R})$ is locally cartesian closed*

Lemma

Let (A, R) be a finitely complete BRO. Then the following are equivalent.

- *(A, R) is relationally complete*
- *$D(A, R)$ is a tripos*
- *$(\widetilde{A, R})$ is a topos*

- Remark: This generalizes a result of Hofstra, who characterized those BCOs A such that DA is a tripos

(Typed) pcas

Definition

Let (I, A, R) be a finitely complete multiform preorder. A *designated truth value* is an element $a \in A_i$ such that $\{(\top, a)\} \in R_{1,i}$

Lemma

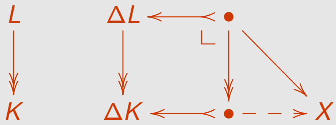
- *The multiform preorders induced by weak typed pcas are precisely the relationally complete functional multiform preorders where all truth values are designated.*
- *Weak (untyped) pcas can be identified with relationally complete functional BROs where all truthvalues are designated.*

Part III
The characterization

Modesty

Definition

Let $\Delta : \mathbb{R} \to \mathbb{X}$ be a regular functor into an exact category.
Call $X \in \mathbb{X}$ *modest* (with respect to Δ), if in any diagram of shape



there exists a mediating arrow,

Characterization

Theorem

Functors of the form $\Delta : \mathbf{Set} \to \mathbf{RT}(\mathcal{A})$ *can be characterized as those functors* $\Delta : \mathbf{Set} \to \mathbb{X}$ *such that*

- $\mathbb{X}$ *is locally cartesian closed and exact*
- $\mathbb{X}(1, -) \dashv \Delta$
- Δ *is regular (redundant in presence of choice)*
- *There exists* $\phi : M \rightarrowtail \Delta A$ *such that*
  - ϕ *is indecomposable projective in* $\mathrm{gl}(\Delta)$
  - M *is modest with respect to* Δ
  - *The subfibration of* $\mathrm{gl}(\Delta)$ *generated by* ϕ *is closed under finite meets*
  - *Every* $X \in \mathbb{X}$ *can be covered like*

$$\begin{array}{c} B \\ \downarrow \\ A \end{array} \qquad \begin{array}{ccccc} \Delta B & \leftarrowtail & \bullet & \twoheadrightarrow & X \\ \downarrow & \lrcorner & \downarrow & & \\ \Delta A & \leftarrowtail & M & & \end{array}$$

Proof

Necessity of conditions

Assume that $\Delta : \mathbf{Set} \to \mathbf{RT}\mathcal{A}$

- The conditions on $\mathbb{X}$ and Δ are well known for realizability toposes
- ϕ is the assembly morphism $: (\mathcal{A}, \iota) \to (\mathcal{A}, =)$ with underlying map the identity, where $\iota(a) = \{a\}$

Sufficiency of conditions

- $\mathbb{X}$ is the fibred cocompletion of the fibration generated by ϕ since ϕ is indecomposable projective and covers everything
- Since the fibration generated by ϕ has a generic predicate, it comes from a BRO (A, R)
- (A, R) is relationally complete since $\mathbb{X}$ is locally cartesian closed
- (A, R) is functional by modesty of M
- All truth-values are designated since $\mathbb{X}(1, -) \dashv \Delta$
- Together, this implies that (A, R) comes from a pca

Thank you for your attention.